THE UNIVERSITY OF ALBERTA

TWO-DIMENSIONAL FLOWS UNDER GRAVITY
WITH A FREE SURFACE

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

by

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EDMONTON, ALBERTA

JULY, 1963

ABSTRACT

This thesis deals with free surface flows of an ideal fluid in a gravity field. The work is restricted to two dimensions and only irrotational flows are considered. Lewy's proof that the free surface of such flows is an analytic curve is given in Chapter Two, while Chapter Three gives an account of some methods, based on Lewy's result, for obtaining the flow corresponding to a particular free surface. Some connections between the various methods and a re-derivation of one of them are given in Chapter Four. Chapter Five contains some examples of flows of this type; most of these involve the flow over a sharp corner of a stream which is uniform and of finite depth at great distances upstream. Such a corner exists in the flows corresponding to a wide class of free surface curves, and some discussion of this phenomenon occurs.

ACKNOWLEDGEMENTS

I would like to thank Dr. J. R. Pounder for suggesting the topic and for his sound guidance and advice. I would also like to thank the Computing Centre for their part in the numerical computations.

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CHAPTER ONE

INTRODUCTION

We shall concern ourselves here with the flow of a fluid with a free surface under the action of gravity. We shall assume that the fluid under consideration is non-viscous and incompressible, and that the motion is both steady and irrotational. By steady flow we mean that the situation at a fixed point in space does not change as time goes on; by irrotational flow we mean that if the velocity at a point in the fluid is expressed as a vector function of position, then this vector has zero curl. We shall also assume that the motion is two-dimensional; that is to say we shall assume that, by a suitable choice of cartesian axes, we can make every physical variable independent of one coordinate. It will suffice, then, to discuss the motion in the plane of the other two coordinates. The term free surface or free streamline will mean any streamline along which the pressure does not vary. This condition is met on the surface of a fluid open to the air. A gravity flow will be taken to mean a flow where the effects of gravity are taken into account. If these effects are neglected we shall speak of a gravity-free flow.

We choose a coordinate system, then, with the y-axis pointing up so that gravity acts in the negative y direction. Denote by $u(x,y)$ and $v(x,y)$ the x- and y-components of the fluid velocity at the point (x,y). The conditions of incompressibility and irrotationality imply that, inside the fluid, the quantities $u(x,y)$ and $-v(x,y)$ satisfy the Cauchy-Riemann equations. Hence the complex velocity ω of the fluid given by

$$\omega = u - i v$$

is an analytic function of the complex variable

$$z = x + i\,y$$

inside the fluid. The speed of the fluid is denoted by V and is given by

$$V = |\omega| \quad .$$

Since ω is analytic, so is its integral, and so the complex potential function

$$f = \varphi(x,y) + i\,\psi(x,y)$$

defined by

$$\frac{df}{dz} = \omega$$

is also an analytic function inside the fluid. This function is arbitrary up to a complex constant and it is not hard to show that the functions $\varphi(x,y)$ and $\psi(x,y)$ are respectively the velocity potential and the stream function for the flow. That is to say the gradient of φ is just the velocity vector function, and the curves ψ = constant are the streamlines of the flow. We choose the arbitrary constant in ψ so that the free surface of the fluid is given by $\psi = 0$.

Since the fluid is incompressible and irrotational then Bernoulli's equation gives that

$$\tfrac{1}{2}\,V^2 + gy + \frac{p}{\rho}$$

is constant throughout the fluid. In this expression p denotes the pressure while ρ is the fluid density. The condition that distinguishes the free surface is that the pressure does not vary there. Hence, if U is a constant with the dimension of velocity, then

$$V^2 + 2gy = U^2$$

is Bernoulli's equation on $\psi = 0$. By suitable choice of origin the constant U can be chosen equal to zero; this is the situation in Chapters Two and Three. In Chapters Four and Five the more general situation is dealt with.

From a practical point of view the most interesting problem is that in which the shape of the bed and the upstream conditions are given, and the free streamline is to be determined. This problem is difficult because it is impossible to compute V on the bed using Bernoulli's equation since the pressure there is not known. On the other hand, the pressure is constant on the free streamline and so the velocity is known if the curve is given. This makes it somewhat easier to determine a flow consistent with that free streamline. The methods dealt with here all solve this latter problem: they assume that the free streamline is a known curve and proceed from this using Bernoulli's equation.

From our point of view, the most important property of the flows which we are considering is that the free streamline is an analytic curve. This was first proved by Lewy [1] in 1952 and the proof is presented in Chapter Two. Using this fact, analytic parametric expressions can be found for f and z which are valid on the free streamline. These are essentially re-statements of Bernoulli's equation and the equation of the free streamline, and we can arrange it so that the parameter involved is real on $\psi = 0$. Since these are analytic relations, their power-series expansions can be used to define analytic functions for complex values of the parameter, and these in turn determine a flow consistent with the given free streamline. This process has been carried out by Lewy [2] and John [3] using quite different formulations of the problem. Milne-Thomson [4] extends the results of John. These methods are presented in Chapter Three and, in

Chapter Four, Lewy's results are re-derived in a different fashion and some connections are made between these results and those of John and Milne-Thomson.

The flows constructed in this way cannot in general be extended away from the free streamline past a certain point. This point corresponds to a singularity in the complex velocity and its location depends on the shape of the free streamline. Any streamline above or through this point can of course be chosen as a bed. The streamline through the singularity often has a sharp corner at that point; the angle and orientation of this corner depend on the flux of fluid down the stream as well as on the shape of the free streamline. This dependence is discussed in Chapter Five, and some illustrative examples are presented there. The first five of these are uniform flows of finite depth far upstream, and they all have a corner of angle $\frac{2\pi}{3}$. This does not mean that other angles are impossible and Example 6, while it does not have the same upstream behavior, exhibits how different angles can occur. In addition there exist flows which have no such singularity: the flow with a straight horizontal free streamline and the flow in the example at the end of Chapter Four are examples of this. However the latter flow has a corner of angle $\frac{2\pi}{3}$ in the free streamline.

CHAPTER TWO

THE ANALYTICITY OF THE FREE STREAMLINE

The method used here was given by Hans Lewy in 1952. [1]. It is used to establish that the free streamline in a flow such as we are considering is an analytic curve. Lewy's result is more general; the relation which he imposes on the boundary of the region of concern is an arbitrary analytic function whereas here we have the explicit relation embodied in Bernoulli's equation. The method, however, is the same.

Bernoulli's equation on $\psi = 0$ reads:

$$\left|\frac{df}{dz}\right|^2 = -2gy . \qquad \psi = 0 \tag{2.01}$$

Here $f = \varphi + i\psi = f(z)$ is an analytic function of $z = x + iy$ in the fluid, i.e. in the region $\psi < 0$. At a point where $\left|\frac{df}{dz}\right| = V \neq 0$, $f = f(z)$ can be inverted to give:

$$z = F(f) = x(\varphi,\psi) + iy(\varphi,\psi) \equiv F(\varphi,\psi) \tag{2.02}$$

which is an analytic function of $f = \varphi + i\psi$ in some neighborhood of a point where $V^2 > 0$. Using equation (2.01) we wish to show that $F(f)$ can be extended analytically across the line $\psi = 0$ wherever $V^2 > 0$.

Now where $V^2 > 0$ we have $\frac{df}{dz} = \left[\frac{dz}{df}\right]^{-1} = \left[\frac{dF}{df}\right]^{-1}$. Hence equation (2.01) becomes:

$$1 = -2gy \left|\frac{dF}{df}\right|^2 . \qquad \psi = 0, V^2 > 0. \tag{2.03}$$

Consider the function

$$G(f) = \overline{F(\bar{f})} . \tag{2.04}$$

On $\psi = 0$ f is real and so $G(f) = \overline{F(f)}$. Hence we have that $2iy = F - \overline{F}$ $= F - G$. Moreover

$$\left|\frac{dF}{df}\right|^2 = \frac{dF}{df}\overline{\left(\frac{dF}{df}\right)} = \frac{dF}{df}\frac{d\overline{F}}{df} = \frac{dF}{df}\frac{dG}{df}$$

on $\psi = 0$, and so equation (2.03) can be written

$$1 = ig(F - G)\frac{dF}{df}\frac{dG}{df} .$$

It follows that

$$\frac{dG}{df} = i\left[g\frac{dF}{df}(G-F)\right]^{-1} \equiv H(G,f). \quad \psi = 0. \tag{2.05}$$

At points where $V^2 > 0$, $F(f)$ is analytic and, in addition, equation (2.01) ensures that y, and hence $G-F$, is non-zero. Hence $H(G,f)$ is an analytic function of both its arguments where $V^2 > 0$.

Inside the fluid where F and G are both analytic functions of f, we can write:

$$\frac{dG}{df} = \frac{\partial}{\partial\varphi}G(\varphi,\psi) = -i\frac{\partial}{\partial\psi}G(\varphi,\psi) , \tag{2.06}$$

$$\frac{dF}{df} = \frac{\partial}{\partial\varphi}F(\varphi,\psi) = -i\frac{\partial}{\partial\psi}F(\varphi,\psi) . \tag{2.07}$$

In order to establish the analyticity of the free streamline we must make some assumptions about x and y. These are that $x(\varphi,\psi)$, $y(\varphi,\psi)$ and $\frac{\partial x}{\partial\varphi}$ exist and are continuous for $\psi \leq 0$. Equation (2.02) implies that x and y are conjugate harmonics with continuous derivatives inside the fluid. Our assumptions together with the mean value theorem imply that $\frac{\partial y}{\partial\psi}$ exists and equals $\frac{\partial x}{\partial\varphi}$ for $\psi \leq 0$. (See Appendix 1). Also equations (2.07) and (2.02) together with (2.03) imply:

$$1 = -2gy\,[(\frac{\partial x}{\partial \varphi})^2 + (\frac{\partial x}{\partial \psi})^2] \tag{2.08}$$

and
$$1 = -2gy\,[(\frac{\partial y}{\partial \varphi})^2 + (\frac{\partial y}{\partial \psi})^2] \tag{2.09}$$

on $\psi = 0$. Hence, also, $\frac{\partial x}{\partial \psi}$ and $\frac{\partial y}{\partial \varphi}$ exist and are continuous in the region $\psi \leq 0$.

Now, at the origin equation (2.04) yields

$$G(0) = \overline{F(0)}\ . \tag{2.10}$$

This condition together with the differential equation (2.05) can be shown (see Appendix 2) to ensure the existence and uniqueness of an analytic function $G(f)$ inside a region where certain conditions on $H(G,f)$ are satisfied. These conditions are:

$$|H(G,f)| < M \tag{2.11}$$

and
$$|\frac{\partial H}{\partial G}| < N\ , \tag{2.12}$$

where M and N are constants.

To show that equation (2.11) is verified we remember that, since $|\frac{dF}{df}| = |\frac{dz}{df}| = 1/V$, equation (2.05) gives

$$|H(G,f)| = 1/g\ |\frac{dF}{dz}|\ |G-F| = V/g\ |G-F|\ .$$

Moreover equation (2.10) yields $|G(0) - F(0)| = |\overline{F}(0) - F(0)| = 2|y|$. Where $V^2 > 0$ equation (2.01) ensures $|y| > 0$ and so $|G-F| > 0$ at the origin. Finally, since y is assumed to be continuous up to, and including the line $\psi = 0$, we have that $|G-F|$ is bounded away from zero in some small neighborhood $\psi \leq 0$ of the origin. This means that the condition (2.11) is met.

Using equation (2.05) once more we obtain

$$\frac{\partial H}{\partial G} = - i/g \frac{dF}{df} (G - F)^2 .$$

This yields $|\frac{\partial H}{\partial G}| < N$ in the same region, and for the same reasons, as above. Hence condition (2.12) is satisfied.

This means that equation (2.05) can be solved for a function $G(f)$ satisfying equation (2.10) which exists, is unique, and is analytic in a small neighborhood $\psi < 0$ of the origin. Its analyticity on $\psi = 0$ will be shown below. It is worthwhile to note that by appropriate choice of the arbitrary constant in the velocity potential φ the origin can be taken at any point on the free streamline, and so the above argument applies at any point where $V^2 > 0$.

We now introduce a function $G(\varphi,\psi)$ by the following construction:

$$G(0,0) = \overline{F(0,0)} , \tag{2.13}$$

$$\frac{\partial}{\partial\psi} G(0,\psi) = -i[g \frac{\partial}{\partial\psi} F(0,\psi) (G(0,\psi)-F(0,\psi))]^{-1} \equiv H_1(G,\psi) \tag{2.14}$$

on $\varphi = 0$, $\psi \leq 0$, and

$$\frac{\partial}{\partial\varphi} G(\varphi,\psi_0) = i[g \frac{\partial}{\partial\varphi} F(\varphi,\psi_0) (G(\varphi,\psi_0)-F(\varphi,\psi_0))]^{-1} \equiv H_2(G,\varphi,\psi_0) \tag{2.15}$$

on $\psi = \psi_0 \leq 0$. Equation (2.14) can be considered as an ordinary differential equation for $G(0,\psi)$. Together with the condition (2.13) it determines a function $G(0,\psi)$ which exists, is unique and continuous (see Appendix 2) in a region $\psi \leq 0$ of the negative ψ axis where conditions comparable to (2.11) and (2.12) are met by $H_1(G,\psi)$. The proof that these conditions are satisfied parallels that given for $H(G,f)$ above. The

continuity follows since F and $\frac{\partial F}{\partial \psi}$ are continuous up to the free surface by hypothesis.

Having thus constructed $G(0,\psi)$, we use this boundary value with equation (2.15). ψ_0 must, of course, be chosen in the region of existence of $G(0,\psi)$. Again (see Appendix 2) the function $G(\varphi,\psi_0)$ defined in this way exists and is unique in a region where conditions analogous to (2.11) and (2.12) are satisfied by $H_2(G,\varphi,\psi_0)$. The region here is $\psi = \psi_0 \leq 0$, $|\varphi| < \epsilon$ for some ϵ . The function $G(0,\psi_0)$ is continuous for $\psi_0 \leq 0$ as is $H_2(G,\varphi,\psi_0)$ by hypothesis. Hence (see Appendix 2) the solution $G(\varphi,\psi_0)$ is continuous in both φ and the parameter ψ_0 for $\psi_0 \leq 0$. It follows that the function $G(\varphi,\psi)$ tends uniformly to $G(\varphi,0)$ as $\psi \to 0$.

We have that $G(\varphi,0)$ is a solution of the differential equation (2.15) with initial value $G(0,0) = \overline{F(0,0)}$. However a solution satisfying this condition is certainly $G(\varphi,0) = \overline{F(\varphi,0)}$, since this substitution changes (2.15) into (2.09). Hence uniqueness requires

$$G(\varphi,0) = \overline{F(\varphi,0)} \; . \tag{2.16}$$

Now if we specialize derivatives of $G(f)$ as constructed from equations (2.05) and (2.10) by the relations (2.06), then on $\varphi = 0$, (2.05) becomes (2.14), and on $\psi = \psi_0 \leq 0$, (2.05) becomes (2.15). The uniqueness of all these solutions then requires that $G(f) = G(\varphi,\psi)$. Moreover, this shows that the analytic function $G(f)$ remains continuous up to the free streamline as $\psi \to 0$. Equation (2.16) now becomes

$$G(\varphi + i0) = \overline{F(\varphi + i0)} \; . \tag{2.17}$$

We now define for $\psi > 0$

$$F(\varphi + i\psi) = G(\varphi - i\psi). \tag{2.18}$$

This is clearly analytic for $\psi > 0$ since $G(\varphi + i\psi)$ is analytic for $\psi < 0$. This definition of the analytic function $F(f)$ for $\psi > 0$ is the same as our previous one on the free streamline by equation (2.17), and so provides the required analytic extension of $F(f)$ across the ψ-axis. This means that $F(f)$ is analytic on the free streamline and so $x(\varphi,\psi)$ and $y(\varphi,\psi)$ are.

Since y is analytic on $\psi = 0$, equation (2.01) shows that V is. Moreover, if we denote distance along the free streamline by s, then this is analytic since

$$s = \int \{1 + (\frac{dy}{dx})^2\}^{\frac{1}{2}} \, dx \quad .$$

Finally, since $\frac{ds}{dt} = V$ where t is time, we have

$$t = \int \frac{ds}{V}$$

and so this quantity too is analytic. The fact that all these quantities are analytic functions of each other will be used extensively in the following.

CHAPTER THREE

METHODS OF OBTAINING FLOWS

In this chapter we outline three methods of obtaining flows when the equation of the free streamline is known. The first two are independent approaches to the problem while the third is an extension of the second.

Section 3.1 The Method of Lewy.

The following method appeared in 1952 in a paper by Hans Lewy [2]. It makes use of the fact that the free streamline is an analytic curve to develop a way of obtaining a flow having a given free streamline. The notation here follows that of Chapter Two.

Whenever $\left|\frac{df}{dt}\right|^2 = V^2 \neq 0$, the equation $f = f(z)$ has an inverse and so, as in equations (2.02) and (2.04), we can define functions $F(f)$ and $G(f)$ as

$$z = z(f) = F(f) \tag{3.01}$$

and

$$G(f) = \overline{F(\bar{f})} \quad . \tag{3.02}$$

On the free streamline, $\psi = 0$ and so f is real. This gives

$$y = -\frac{i}{2}[F(f) - G(f)] \quad . \qquad \psi = 0 \tag{3.03}$$

Also, since $\left|\frac{dF}{df}\right|^2 = \frac{dF}{df}\frac{dG}{df}$ on $\psi = 0$, we can write the Bernoulli equation (2.03) as

$$1 = ig(F - G)\frac{dF}{df}\frac{dG}{df} \quad . \qquad \psi = 0 \tag{3.04}$$

We now define a quantity

$$\lambda(f) = \frac{i}{2}[F(f) - G(f)] \quad . \tag{3.05}$$

This is a real quantity on $\psi = 0$ and, comparing equation (3.03) and (3.05), we see that on the free streamline

$$\lambda = -y \quad . \qquad \psi = 0 \tag{3.06}$$

Using equation (3.05) in equation (3.04) yields

$$1 = 2g\lambda \frac{dF}{df}\frac{dG}{df} \quad . \qquad \psi = 0 \tag{3.07}$$

We now solve equations (3.05) and (3.07) for $F(f)$ in terms of the function $\lambda(f)$. If we differentiate equation (3.05) with respect to f, solve the result for $\frac{dG}{df}$ and substitute this into equation (3.07) we get, after some simplification that

$$\left(\frac{dF}{df}\right)^2 + 2i\frac{d\lambda}{df}\frac{dF}{df} - \frac{1}{2g\lambda} = 0 \quad .$$

This gives $$\frac{dF}{df} = -i\frac{d\lambda}{df} \pm \left[\frac{1}{2g\lambda} - \left(\frac{d\lambda}{df}\right)^2\right]^{\frac{1}{2}} \quad .$$

On the free streamline λ and f are both real. Hence, since $F = x+iy$, the real part of this is

$$\frac{dx}{d\varphi} = \pm\left[\frac{1}{2g\lambda} - \left(\frac{d\lambda}{d\varphi}\right)^2\right]^{\frac{1}{2}}$$

provided $\frac{1}{2g\lambda} \geq \left(\frac{d\lambda}{d\varphi}\right)^2$. In order that x increase with φ we choose the positive root and so

$$F(f) = z(f) = -i\lambda + \int \left[\frac{1}{2g\lambda} - \left(\frac{d\lambda}{df}\right)^2\right]^{\frac{1}{2}} df \, . \tag{3.08}$$

When $\lambda(f)$ is known, equation (3.08) gives z as a function of $f = \varphi+i\psi$. In theory this can be inverted where V^2 is not singular to yield f as a function of z and hence give the velocity potential and stream function.

If we remember that $\frac{1}{V^2} = \left|\frac{dF}{df}\right|^2 = \frac{dF}{df}\frac{dG}{df}$, then equation (3.07) becomes

$$V^2 = 2g\lambda . \qquad \psi = 0 \qquad (3.09)$$

This gives some conditions on the function $\lambda(f)$. Equation (3.06) implies that $\lambda(f)$ is real on $\psi = 0$, that is it is real for real arguments. In addition equation (3.09) gives

$$\lambda = 0 \quad \text{if} \quad V^2 = 0 ,$$

$$\text{and} \quad \lambda > 0 \quad \text{if} \quad V^2 > 0 . \qquad \psi = 0 \qquad (3.10)$$

Now $z(f) = x(f) + iy(f)$ where $x(f)$ and $y(f)$ are real for real arguments. Hence $\frac{dz}{df} = \frac{dx}{df} + i\frac{dy}{df}$ and so

$$\left|\frac{dz}{df}\right|^2 = \frac{1}{V^2} = \left(\frac{dx}{df}\right)^2 + \left(\frac{dy}{df}\right)^2 \geq \left(\frac{dy}{df}\right)^2 = \left(\frac{d\lambda}{df}\right)^2 \quad \text{on} \quad \psi = 0.$$

Hence
$$\frac{1}{V^2} = \frac{1}{2g\lambda} \geq \left(\frac{d\lambda}{df}\right)^2 . \qquad (3.11)$$

If $\lambda(f)$ satisfies equations (3.10) and (3.11), then (3.08) will be a valid relationship on the free streamline. Since all the functions involved are analytic, this equation, valid for real values of f, will remain valid for complex values. The flow given by the resulting complex potential will have the property that the pressure is constant on the streamline given by setting $\psi = 0$. It will thus represent some gravity flow with this as free streamline. This, in essence, is the type of argument which underlies all the methods of this chapter. It depends heavily on the fact that the free streamline is necessarily an analytic curve. This, of course, was proved in Chapter Two.

When the free surface is prescribed, this method gives the corresponding flow under gravity. Since the free surface is analytic, there exists a relation between its slope and its elevation. Suppose this is

$$\frac{dx}{dy} = \sigma(-y) = \sigma(\lambda) \quad . \tag{3.12}$$

Here $\sigma(\lambda)$ is real for real arguments and analytic. Now, on $\psi = 0$, λ and f are real and $\frac{1}{2g\lambda} \geq (\frac{d\lambda}{df})^2$ by equation (3.11). Hence equation (3.08) gives

$$x = \int [\frac{1}{2g\lambda} - (\frac{d\lambda}{df})^2]^{\frac{1}{2}} \, df \, .$$

Since on $\psi = 0$, $\lambda = -y$, this yields

$$\frac{dx}{dy} = \sigma(\lambda) = - [\frac{1}{2g\lambda} - (\frac{d\lambda}{df})^2]^{\frac{1}{2}} \frac{df}{d\lambda} \quad ,$$

and so

$$- \sigma(\lambda) \frac{d\lambda}{df} = [\frac{1}{2g\lambda} - (\frac{d\lambda}{df})^2]^{\frac{1}{2}} \, . \tag{3.13}$$

Solution of this and integration yields

$$f = \int [2g\lambda(1+\sigma^2(\lambda))]^{\frac{1}{2}} \, d\lambda \, . \tag{3.14}$$

Substitution of equation (3.13) into equation (3.08) gives

$$z = - i\lambda - \int \sigma(\lambda) d\lambda \quad . \tag{3.15}$$

Equations (3.14) and (3.15) give f and z in terms of the parameter λ when $\sigma(\lambda)$ is known. Again they are valid for complex λ since all the functions concerned are analytic and they are consistent with Bernoulli's equation on $\psi = 0$. They provide the most direct method known to the author of obtaining the flow having a particular free streamline, and will be re-derived in a slightly more general form in the next chapter. The extent of the motion so defined depends on the properties of $\sigma(\lambda)$ for complex values of λ. To quote Lewy: "[the motion] cannot be expected to depend continuously on the shape of the free surface since an approximation to $\sigma(\lambda)$ for real λ by other analytic functions need not be an approximation for complex values of λ, and it is these that determine the motion

below (or above) the surface." This remark will be validated somewhat in Chapter Five by some examples all of which display a singularity in the flow.

Equation (3.14) can be re-written to give f directly as a function of z provided $\lambda(z)$ can be determined by inverting equation (3.15). Differentiation of equation (3.15) gives

$$\frac{dz}{d\lambda} = -i - \sigma .$$

Hence

$$[1+\sigma^2]\, d\lambda = [\frac{1+\sigma^2}{(-i-\sigma)^2}]^{\frac{1}{2}}\, dz$$

$$= [\frac{i-\sigma}{-i-\sigma}]^{\frac{1}{2}}\, dz$$

$$= [1+2i\,\frac{d\lambda}{dz}]^{\frac{1}{2}}\, dz .$$

This gives, using equation (3.14), that

$$f = \int [2g\lambda(1+2i\,\frac{d\lambda}{dz})]^{\frac{1}{2}}\, dz . \qquad (3.16)$$

This gives f in terms of z if $\lambda(z)$ is known.

Section 3.2 The Method of John.

In a paper in 1953, Fritz John [3] outlined a method of obtaining non-steady gravity flows. We will restrict ourselves to the case of steady flows. His approach, like that of Lewy, consists of writing down analytic parametric expressions for f and z which are valid on the free streamline. He chooses as parameter the time taken by a particle to travel down the free streamline from some fixed point to its present location. This parameter is real on the free streamline and John analytically extends f and z into the fluid by letting it assume complex values.

Suppose then we are given real parametric equations for the free

streamline, the parameter being the time t. That is, suppose $x(t)$ and $y(t)$ give the position of a particle on the free streamline at time t. These are analytic functions of t as was shown at the end of Chapter Two, and we suppose that they are real for real arguments. If we define

$$z(t) = x(t) + iy(t) \tag{3.17}$$

then $z(t)$ gives the free streamline as an analytic curve in the complex plane. We assume t is a real quantity on $\psi = 0$. Now:

$$\frac{dz}{dt} = \frac{dx}{dt} + i\,\frac{dy}{dt} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} \exp\left[i \tan^{-1}\left(\frac{dy}{dx}\right)\right]$$

$$= \frac{ds}{dt} \exp\left[i \tan^{-1}\left(\frac{dy}{dx}\right)\right] ,$$

where $ds = \sqrt{dx^2+dy^2}$ is an increment of arc length measured along the free streamline. Hence $\frac{ds}{dt}$ is the speed of the particle along the free streamline and, if we denote the x- and y- components of the velocity of the particle by u and v respectively, we have on the free streamline:

$$\frac{dz}{dt} = u + iv \quad . \qquad\qquad \psi = 0 \tag{3.18}$$

We will not call this the complex velocity of the particle but will reserve this title for the quantity $\omega = \frac{df}{dz} = u-iv$. It can be seen in a similar way that the acceleration of a particle at the point $z(t)$ on the free streamline is given by $\frac{d^2z}{dt^2}$.

Using this formulation of the problem we can now obtain expressions for $z(t)$ and the complex potential $f(t)$ inside the fluid. We first derive a differential equation for $z(t)$ by considering the equation of motion of a particle at the point $z(t)$ on the free streamline. It is acted upon by two external forces: gravity and the pressure of neighboring particles. If gravity acts in the negative y-direction the equation of

motion can be written as

$$\frac{d^2z}{dt^2} = -ig + \frac{\nabla P}{\rho}$$

where ∇P represents the pressure gradient and ρ is the density of the fluid. Now since the pressure is constant along the free streamline, ∇P is normal to it. We also have that $i\frac{dz}{dt}$ is a normal vector to the free streamline since $\frac{dz}{dt}$ represents the velocity there. It follows that ∇P and $i\frac{dz}{dt}$ are proportional; the proportionality factor must be real but need not be constant and could vary with time. Call it $\frac{d}{dt}(\chi(t))$. Then:

$$\frac{d^2z}{dt^2} + ig = i\,\frac{d\chi}{dt}\,\frac{dz}{dt} \quad . \qquad \psi = 0 \tag{3.19}$$

The function $\frac{d\chi}{dt}$ must be real for real arguments so as to preserve the vector character of equation (3.19) on the free streamline (where t is real). The equation can be solved to yield

$$z(t) = -ig \int^{t} e^{i\chi(\lambda)} \left[\int^{\lambda} e^{-i\chi(\mu)}\, d\mu\right] d\lambda. \tag{3.20}$$

If $\chi(t)$ is given, equation (3.20) gives the free streamline for some gravity flow.

The complex potential $f(t)$ can now be determined from $z(t)$. The complex velocity is

$$\omega = \frac{df}{dz} = u - iv = \overline{\left(\frac{d}{dt}\, z(t)\right)}$$

on the free streamline, using equation (3.18). Since t is real on $\psi = 0$, $t = \bar{t}$ and this can be written

$$\frac{df}{dz} = \frac{d}{dt}\,\overline{[z(\bar{t})]} \quad . \qquad \psi = 0 \; .$$

It follows from the analyticity of $z(t)$ that $\frac{d}{dt}\,\overline{[z(\bar{t})]}$ will be an analytic function of t even if t is allowed to assume complex values.

Finally, since $\frac{df}{dz} = \frac{df}{dt}\frac{dt}{dz} = \frac{df}{dt}(\frac{dz}{dt})^{-1}$, we have

$$\frac{df}{dt} = \frac{d}{dt}[z(t)]\,\frac{d}{dt}[\overline{z(\bar{t})}] \ . \tag{3.21}$$

This is a valid statement with t real and since $z(t)$ is analytic it remains valid when t assumes complex values. Hence if we define $f(t)$ by (3.21) for complex t the result will be analytic in t and hence in z and will thus represent the complex potential for some gravity flow consistent with the free streamline given by $z(t)$.

John uses equation (3.19) to show that if

$$y = \eta(x) \tag{3.22}$$

is the equation of the free streamline, then x can be obtained as a function of t by solving a second order ordinary differential equation. On $\psi = 0$, $x(t)$ and $y(t)$ are real and so equating real and imaginary parts of equation (3.19) leads to the equations

$$\frac{d^2x}{dt^2} = -\frac{d\chi}{dt}\frac{dx}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} + g = \frac{d\chi}{dt}\frac{dx}{dt} \ .$$

If we eliminate $\frac{d\chi}{dt}$ and write $\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}\frac{d\eta}{dx} + (\frac{dx}{dt})^2\frac{d^2\eta}{dx^2}$ then we obtain

$$[1 + (\frac{d\eta}{dx})^2]\frac{d^2x}{dt^2} + \frac{d\eta}{dx}\frac{d^2\eta}{dx^2}(\frac{dx}{dt})^2 + g\frac{d\eta}{dx} = 0. \tag{3.23}$$

This is non-linear except where the free streamline is a straight line.

Section 3.3 Tangent Flows of Milne-Thomson.

In 1959 L. M. Milne-Thomson [4] gave the following extension of the method of John. He obtains a gravity flow which reduces to a known gravity-free flow if g approaches zero.

Suppose we are given the real functions $x_0(t)$ and $y_0(t)$ of time which give the position of a prticle on the free streamline of a gravity-free flow at time t. If we write

$$z_0(t) = x_0(t) + i\, y_0(t) , \tag{3.24}$$

then Bernoulli's equation on $\psi = 0$ reads

$$\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2 = \frac{dz_0}{dt}\,\overline{\frac{dz_0}{dt}} = U_0^2 \tag{3.25}$$

since $g = 0$. U_0 is a constant which represents the velocity of the fluid on the free streamline of the gravity-free flow. As we have defined it, $z_0(t)$ must satisfy equation (3.19) with $g = 0$. That is

$$\frac{d^2z_0}{dt^2} = i\,\frac{d\chi}{dt}\,\frac{dz_0}{dt} . \tag{3.26}$$

This gives $\frac{d\chi}{dt} = -i\,\frac{d}{dt}\,[\ell n(\frac{dz_0}{dt})]$. Since equation (3.25) implies that U_0 is the fluid velocity on $\psi = 0$ in the zero-gravity case, we can write $\frac{dz_0}{dt} = U_0 e^{i\theta_0}$ and so

$$\frac{d\chi}{dt} = -i\,\frac{d}{dt}[\ell n U_0 + i\,\theta_0] = \frac{d\theta_0}{dt} = \frac{d}{dt}\,\tan^{-1}\left(\frac{dy_0}{dx_0}\right) .$$

This is real when t is real and so it follows that this value of $\frac{d\chi}{dt}$ can be used in equation (3.19) to give a gravity flow. In fact

$$\frac{d^2z}{dt^2} + i\,g = \left[\frac{d^2z_0}{dt^2}\Big/\frac{dz_0}{dt}\right]\frac{dz}{dt} \tag{3.27}$$

gives a gravity flow which reduces to $z(t) = Az_0(t) + B$ when $g = 0$. Moreover this flow contains an extra arbitrary constant - the skin speed U_0 of the gravity-free case. This flow is called the tangent flow to $z_0(t)$.

If we use equation (3.25), then equation (3.27) can be written

$$\frac{d}{dt}\left[\frac{dz}{dt}\Big/\frac{dz_0}{dt}\right] = \frac{-ig}{U_0^2}\,\overline{\frac{dz_0}{dt}} \quad . \tag{3.28}$$

This has the property that it has the same form if another parameter τ is used in place of t. If $z_0(\tau) = x_0(\tau) + i\, y_0(\tau)$ and if τ is real on the free streamline, then equation (3.28) becomes

$$\frac{d}{d\tau}\left[\frac{dz}{d\tau}\Big/\frac{dz_0}{d\tau}\right] = \frac{-ig}{U_0^2}\,\overline{\frac{dz_0}{d\tau}} \quad .$$

This can be solved in the form

$$z = A\, z_0 + B - \frac{ig}{U_0^2}\int \overline{z_0}\; dz_0 \quad , \tag{3.29}$$

where A and B are constants of integration. The speed V on the free streamline of the gravity flow is obtained as follows: From equation (3.29) we have

$$\frac{dz}{dt} = \frac{dz_0}{dt}\left[A - \frac{ig}{U_0^2}\,\overline{z_0}\right] \quad .$$

Using equation (3.25) it follows that

$$V^2 = \frac{dz}{dt}\,\overline{\frac{dz}{dt}} = U_0^2\,|A|^2 - \mathrm{Im}\,(Az_0) + \frac{g^2}{U_0^2}\,|z_0|^2 \quad . \tag{3.30}$$

Equation (3.29) yields the following expressions for x and y. If $B = b_1 + ib_2$, we have

$$x = \mathrm{Re}(Az_0) + b_1 + \frac{g}{U_0^2}\int (x_0 dy_0 - y_0 dx_0)$$

and

$$y = \mathrm{Im}\,(Az_0) + b_2 - \frac{g}{2U_0^2}\,|z_0|^2 \quad .$$

The second of these and equation (3.30) imply

$$V^2 + 2gy = U_0^2\,|A|^2 + 2gb_2 \quad .$$

This has the form of equation (2.01) if we choose the origin of coordinates such that $b_2 = -\frac{U_0|A|^2}{2g}$. We can choose $b_1 = 0$ and so get

$$x = \mathrm{Re}(Az_0) + \frac{g}{U_0^2} \int (x_0\, dy_0 - y_0\, dx_0) \tag{3.31}$$

and

$$y = \mathrm{Im}\,(Az_0) - \frac{g}{2U_0^2} |z_0|^2 - \frac{U_0^2\,|A|^2}{2g} \;. \tag{3.32}$$

If we choose the free streamline in the gravity-free case to be a circle, then equations (3.31) and (3.32) give the example done in John's paper [3] where the free streamline is a trochoid. In addition spiral shaped curves can lead to monotonic tangent free streamlines. In fact the shape of the tangent free streamline seems in general to bear little resemblance to that of the free streamline in the gravity free case. For this reason it would seem that the name Tangent Flow is a rather inept one.

CHAPTER FOUR

RELATIONS BETWEEN THESE FORMULATIONS

It is evident that the above are all different formulations of the solution to the same problem. To indicate precisely the connection between them we first reformulate the problem and obtain Lewy's results in a slightly more general form, and then derive the connections between these and the results of John and Milne-Thomson.

If we choose the quantity λ such that

$$\lambda = -y \qquad\qquad \psi = 0 \tag{4.01}$$

on the free streamline, then the Bernoulli equation can be written

$$V^2 = U^2 + 2g\lambda \tag{4.02}$$

there, where U is a constant with the dimension of velocity. Lewy found it convenient to choose it equal to zero but we leave it arbitrary. Equation (4.02) requires that $\lambda \geq -\dfrac{U^2}{2g}$ on the free streamline.

We assume that we are given a real analytic curve which we want to be the free streamline of a gravity flow. Suppose it has the form

$$x = x(\lambda) = x(-y) \ . \tag{4.03}$$

If we use equation (4.01) this curve is given by

$$z = -i\lambda + x(\lambda) \tag{4.04}$$

in the plane of the complex variable $z = x + iy$. As in Lewy's paper $x(\lambda)$ is real for real arguments, and so equation (4.04) represents the free streamline if λ is real and $\lambda \geq -\dfrac{U^2}{2g}$.

We want to construct an analytic velocity function ω with the property that $\omega \bar{\omega} = V^2 = U^2 + 2g\lambda$ on the free streamline in accordance with equation (4.02). Since λ is to be real on the free streamline, the function

$$\omega = (U^2 + 2g\lambda)^{\frac{1}{2}} e^{i\theta(\lambda)}$$

satisfies this condition if $\theta(\lambda)$ is an analytic function which is real for real arguments $\lambda \geq -\frac{U^2}{2g}$.

Now suppose f is the complex potential. Then since $\omega = \frac{df}{dz}$, we have using equations (4.04) and (4.05) that

$$\frac{df}{d\lambda} = \frac{df}{dz}\frac{dz}{d\lambda} = \omega(-i + \frac{dx}{d\lambda}) .$$

Hence:

$$f = \int (U^2+2g\lambda)^{\frac{1}{2}} (\cos\theta+i\sin\theta)(-i+\frac{dx}{d\lambda})d\lambda = \varphi(\lambda)+i\psi(\lambda) .$$

Now if we call the free streamline $\psi(\lambda) = 0$, then since λ, $\theta(\lambda)$ and $x(\lambda)$ are all real there, we can write

$$\psi(\lambda) = \int (U^2+2g\lambda)^{\frac{1}{2}} (\sin\theta \frac{dx}{d\lambda} - \cos\theta)d\lambda = 0$$

on the free streamline. This requires

$$\frac{dx}{d\lambda} = \cot(\theta) , \quad \text{or} \quad \theta = \cot^{-1}(\frac{dx}{d\lambda}) . \tag{4.06}$$

This clearly gives $\bar{\omega}$ the direction of the tangent to our curve which is as it should be.

Following Lewy we now define the analytic function

$$\sigma(\lambda) = -\frac{dx}{d\lambda} . \tag{4.07}$$

Again $\sigma(\lambda)$ is real if λ is real and $\lambda \geq -\frac{U^2}{2g}$. Using equation (4.07) we can write equation (4.04) as

$$z = -i\lambda - \int \sigma(\lambda)d\lambda \tag{4.08}$$

which is equation (3.15) of Lewy. We have also

$$e^{i\theta(\lambda)} = e^{i\cot^{-1}(-\sigma)} = \left(\frac{\sigma-i}{\sigma+i}\right)^{\frac{1}{2}} . \tag{4.09}$$

Hence:

$$\omega = (U^2+2g\lambda)^{\frac{1}{2}} \left(\frac{\sigma-i}{\sigma+i}\right)^{\frac{1}{2}} , \tag{4.10}$$

and

$$f = \int (U^2+2g\lambda)^{\frac{1}{2}} (1+\sigma^2)^{\frac{1}{2}} d\lambda . \tag{4.11}$$

Equation (4.11) reduces to equation (3.14) of Lewy if $U = 0$ and so provides a slight generalization of that equation. The factor U^2 in equations (4.10) and (4.11) allows us to construct flows which have the property that they are uniform streams of finite depth at great distances upstream. In addition the factor $(\sigma+i)^{\frac{1}{2}}$ in the denominator of equation (4.10) shows that a singularity in the flow is likely to occur where $\sigma = -i$. This branch point in the velocity determines a natural bound on the extent to which the flow can be analytically continued away from the free streamline, a phenomenon which is in keeping with the remark of Lewy's quoted in Chapter Three. Some examples of this will be seen in Chapter Five.

We next obtain expressions for the quantities t and χ of John in terms of λ and $\sigma(\lambda)$. We begin by obtaining equation (3.21) using a method different from that used by John.

If we write $\omega = u - iv$, then we can write equation (4.02) in the form

$$U^2 + 2g\lambda = \omega\,\bar{\omega} = \omega(u + iv) .$$

Following John we define a new parameter, the time t, on the free streamline by

$$\frac{dz}{dt} = u + iv = \bar{\omega} . \tag{4.12}$$

We thus have $\omega(u+iv) = \frac{df}{dz} \cdot \frac{dz}{dt} = \frac{df}{dt}$ and so the Bernoulli relation on the free streamline becomes

$$\frac{df}{dt} = U^2 + 2g\lambda \quad . \qquad \psi = 0 \tag{4.13}$$

This implies that the time t is a real quantity on the free streamline since f and λ are real there. Using these results we can write

$$\frac{df}{dt} = \omega \bar{\omega} = \frac{dz}{dt} \overline{\frac{dz}{dt}}$$

on the free streamline. Since t is real there, then if we refer z to an argument t we can write $z(t) = z(\bar{t})$ and so $\overline{\frac{dz}{dt}} = \frac{d}{dt} \overline{z(\bar{t})}$. Finally

$$\frac{df}{dt} = \frac{d}{dt} z(t) \frac{d}{dt} \overline{z(\bar{t})} \quad , \tag{4.14}$$

which is equation (3.21) of John.

The connection between the quantities λ and t on the free streamline is found from equations (4.11) and (4.13). We have

$$\frac{df}{dt} = U^2 + 2g\lambda = \frac{df}{d\lambda} \frac{d\lambda}{dt} = (U^2 + 2g\lambda)^{\frac{1}{2}} (1 + \sigma^2)^{\frac{1}{2}} \frac{d\lambda}{dt} \quad .$$

Hence:

$$t = \int \left(\frac{1 + \sigma^2}{U^2+2g\lambda}\right)^{\frac{1}{2}} d\lambda \tag{4.15}$$

which is an analytic function of λ on the free streamline.

We are now in a position to write down χ as a function of λ and $\sigma(\lambda)$. John writes the equation of motion of a particle on the free streamline as

$$\frac{d^2z}{dt^2} + ig = i \frac{d\chi}{dt} \frac{dz}{dt}$$

where the function χ is real on the free streamline. We can write

$$\frac{d^2z}{dt^2} = \frac{d}{dz}\left(\frac{dz}{dt}\right)\frac{dz}{dt} = \bar{\omega}\,\frac{d\bar{\omega}}{dz}$$

using equation (4.12), and so the equation of motion becomes

$$\bar{\omega}\,\frac{d\bar{\omega}}{dz} + ig = i\,\frac{d\chi}{dz}\,(\bar{\omega})^2 \quad .$$

Some algebra and integration gives

$$\chi = -i\,\ell n(\bar{\omega}) + \int \frac{g dz}{(\bar{\omega})^2} \quad .$$

Now from equation (4.08), $dz = (-i-\sigma)d\lambda$ and so, using equation (4.05),

$$\chi = -i[-i\theta+\tfrac{1}{2}\ell n(U^2+2g\lambda)] + \int g\left(\frac{-i-\sigma}{U^2+2g\lambda}\right)e^{-2i\theta}\,dy \ .$$

Using equation (4.09) we obtain:

$$\chi = -\theta - \frac{i}{2}\,\ell n(U^2+2g\lambda) - \int \frac{g(i+\sigma)}{U^2+2g\lambda}\left(\frac{\sigma-i}{\sigma+i}\right)d\lambda \quad .$$

Finally,

$$\chi = -\theta - \int \frac{g\sigma d\lambda}{U^2+2g\lambda} \quad . \tag{4.16}$$

Using equations (4.06) and (4.07) we can write this as

$$\chi = \cot^{-1}(\sigma) - \int \frac{g\sigma d\lambda}{U^2+2g\lambda} \quad . \tag{4.17}$$

This again is an analytic function of λ in the free streamline.

We now establish an analytic relationship between z_0 of Milne-Thomson and the quantities λ and χ. If we are given an analytic function of time, $z_0(t)$, satisfying

$$\frac{dz_0}{dt}\,\overline{\frac{dz_0}{dt}} = U^2 \quad , \tag{4.18}$$

then

$$z = Az_0 + B - \frac{ig}{U^2} \int \overline{z_0} \; dz_0 \tag{4.19}$$

is a gravity flow. Equation (4.18) is just equation (3.25) with U_0 chosen equal to U as defined by equation (4.01). Differentiating and using equation (4.12):

$$\frac{dz}{dt} = (A - \frac{ig}{U^2} \overline{z_0}) \frac{dz_0}{dt} = \bar{\omega} \quad .$$

Hence:

$$\omega = (\bar{A} + \frac{ig}{U^2} z_0)\overline{\frac{dz_0}{dt}} = U^2(\bar{A} + \frac{ig}{U^2} z_0) \Big/ \frac{dz_0}{dt} \quad ,$$

using equation (4.18). This gives

$$\frac{dt}{\omega} = \frac{dz_0}{U^2\bar{A}+igz_0} \quad .$$

If we use equation (4.13), then

$$\omega = \frac{df}{dz} = \frac{df}{dt}\frac{dt}{dz} = (U^2 + 2g\lambda)\frac{dt}{dz} \quad ,$$

and so

$$\frac{dt}{\omega} = \frac{dz}{U^2+2g\lambda} = \frac{dz_0}{U^2\bar{A}+igz_0} \quad .$$

Since $dz = (-i-\sigma)d\lambda$ from equation (4.08), then we can write this as

$$\frac{dz_0}{\bar{A} + \frac{ig}{U^2} z_0} = \frac{-(i+\sigma)\; d\lambda}{1 + \frac{2g}{U^2} \lambda}$$

where the denominators are dimensionless. Integration gives:

$$\frac{U^2}{ig} \ln(\bar{A} + \frac{ig}{U^2} z_0) = - \frac{iU^2}{2g} \ln(1 + \frac{2g}{U^2} \lambda) - \int \frac{\sigma \; d\lambda}{1 + \frac{2g}{U^2} \lambda} \quad .$$

This reduces to

$$\ln\left[\frac{\bar{A} + \frac{ig}{U^2} z_0}{(1 + \frac{2g}{U^2} \lambda)^{\frac{1}{2}}}\right] = -i \int \frac{g\sigma \; d\lambda}{U^2 + 2g\lambda} \quad .$$

If we use equation (4.16) to eliminate the integral on the right we obtain:

$$\bar{A} + \frac{ig}{U^2} z_0 = (1 + \frac{2g\lambda}{U^2}) e^{i(\chi+\theta)} \quad .$$

Finally, using equation (4.05), we have

$$\bar{A} + \frac{ig}{U^2} z_0 = \frac{\omega}{U} e^{i\chi} \quad . \tag{4.20}$$

We can now show that the constant A in equation (4.19) can be chosen to be zero without loss of generality. This merely amounts to choosing a different function z_0. To see this suppose we choose a function z_1 given by

$$\frac{ig}{U^2} z_1 = \bar{A} + \frac{ig}{U^2} z_0 \quad .$$

We have that $\frac{dz_1}{dt} = \frac{dz_0}{dt}$ and so equation (4.18) is satisfied by z_1. Also we have

$$\int \overline{z_0}\, dz_0 = \int \overline{z_1}\, dz_1 - \frac{iU^2}{g} A\, z_1 \; .$$

Hence, from equation (4.19) we have

$$\begin{aligned} z &= A \left[z_1 + i \frac{U^2}{g} \bar{A} \right] + B - \frac{ig}{U^2}{}_0 \int \overline{z_1}\, dz_1 - Az_1 \\ &= B - \frac{ig}{U^2} \int \overline{z_1}\, dz_1 + i\frac{U^2|A|^2}{g} \quad . \end{aligned}$$

This is just equation (4.19) with $A = 0$, $z_0 = z_1$, and the origin of z translated vertically. Hence, the simplest formulation of the tangent flows is:

$$\left.\begin{aligned} &\frac{dz_0}{dt} \overline{\frac{dz_0}{dt}} = U^2 \\ &z = C - \frac{ig}{U^2} \int \overline{z_0}\, dz_0 , \\ &z_0 = - \frac{iU\omega}{g} e^{i\chi} \; , \end{aligned}\right\} \tag{4.21}$$

where C is a complex constant.

Equation (4.20) shows that z_0 is an analytic function of λ since ω and χ are. Also, every flow determines a $\sigma(\lambda)$ which is analytic for $\lambda > -\frac{U^2}{2g}$. This in turn yields a z_0. Conversely every z_0 determines some flow. Hence, every flow is a tangent flow.

Equation (4.21) admits another conclusion. If we replace z_0 by $z_0\theta^{i\alpha}$ where α is a real constant, then the tangent flow with free streamline given by z is unchanged. Hence the orientation of a given gravity free flow leaves the resulting tangent flow unchanged, and so we can say that every flow is tangent to an infinity of flows.

Since the function χ appears in a differentiated form in equation (3.19), then two functions χ_1 and χ_2 which differ by a constant will lead to the same gravity flow. The function σ, however, has no such arbitrary character.

We now cite an example which has a free surface consisting of two straight lines meeting in an angle of $\frac{2\pi}{3}$. We will use the solution embodied in equations (4.02), (4.08) and (4.11). However we choose $U = 0$ so as to simplify the notation. This means that the corner in the free streamline will occur at $\lambda = 0$ since it is a stagnation point. The rest of the free surface will correspond to $\lambda > 0$, that is to say $y < 0$.

If β is a constant acute angle, and if we choose

$$\sigma = -\cot\beta \quad ,$$

then equation (4.08) with $U = 0$ gives

$$z = (-i + \cot\beta)\lambda = \frac{\lambda e^{-i\beta}}{\sin\beta} \quad ,$$

and equation (4.11) gives

$$f = \int (2g\lambda)^{\frac{1}{2}} (1+\cot^2\beta)^{\frac{1}{2}} d\lambda = \frac{2}{3}\frac{\sqrt{2g}}{\sin\beta} \lambda^{3/2} .$$

Eliminating λ these give

$$f = \frac{2}{3} (2\theta \sin\beta)^{\frac{1}{2}} e^{\frac{3i\beta}{2}} z^{3/2} .$$

If $z = re^{i\theta}$, then

$$f = \frac{2}{3} r^{3/2} (2g \sin\beta)^{\frac{1}{2}} e^{i3/2(\theta+\beta)} .$$

The free streamline corresponds to $\psi = 0$ and this occurs when

$$\frac{3}{2} (\theta + \beta) = n \pi$$

$$\theta = -\beta + \frac{2n\pi}{3} .$$

Since $y = r \sin \theta$, the condition that $y < 0$ implies that $-\pi < \theta < 0$. Hence only two acceptable values of θ are possible: choose them to correspond to $n = 0$ and $n = -1$. These give

$$\theta = -\beta \quad , \qquad \theta = -(\beta + \frac{2\pi}{3}) .$$

Now $\omega = \frac{df}{dz} = (2g \sin\beta)^{\frac{1}{2}} e^{\frac{3i\beta}{2}} z^{\frac{1}{2}}$. Hence

$$V^2 = \omega \bar{\omega} = 2g \sin\beta \; |z| = 2g \sin \beta \; r .$$

With $U = 0$, Bernoulli's equation (4.02) reads $V^2 = -2gy$ on the free streamline. In the present example this gives

$$2g \sin\beta \; r = - 2gy = -2g \sin \theta \; r .$$

The portion of the free streamline given by $\theta = -\beta$ satisfies this. The portion given by $\theta = -(\beta + \frac{2\pi}{3})$ does so if

$$\sin\beta = \sin(\beta + \frac{2\pi}{3}) .$$

This implies $\qquad \beta = \frac{\pi}{6}$.

Hence $\qquad f = \frac{2\sqrt{g}}{3} e^{i\frac{\pi}{4}} z^{3/2} = \frac{2\sqrt{g}}{3} r^{3/2} e^{i(\frac{3\theta}{2} + \frac{\pi}{4})}$,

and so the stream function is given by

$$\psi = \frac{2\sqrt{g}}{3} r^{3/2} \sin(\frac{3\theta}{2} + \frac{\pi}{4}) .$$

If we set $\theta = -\frac{\pi}{2} + \xi$, then $\sin(\frac{3\theta}{2} + \frac{\pi}{4}) = -\cos\frac{3\xi}{2}$. Thus

$$\psi = -\frac{2\sqrt{g}}{3} r^{3/2} \cos(\frac{3\xi}{2}) .$$

The two free streamlines correspond to $\xi = \pm\frac{\pi}{6}$. The streamlines correspond to $\psi = -\psi_o$, where $\psi_o > 0$. Their equation is, then,

$$r = (\frac{3\psi_o}{2\sqrt{g}})^{2/3} \sec^{2/3}(\frac{3\xi}{2}) .$$

The highest point on any of these curves occurs when $\xi = 0$; that is when $\theta = -\frac{\pi}{2}$. They are symmetrical about this point. A sketch of the flow appears in figure 4.01 with a typical streamline included.

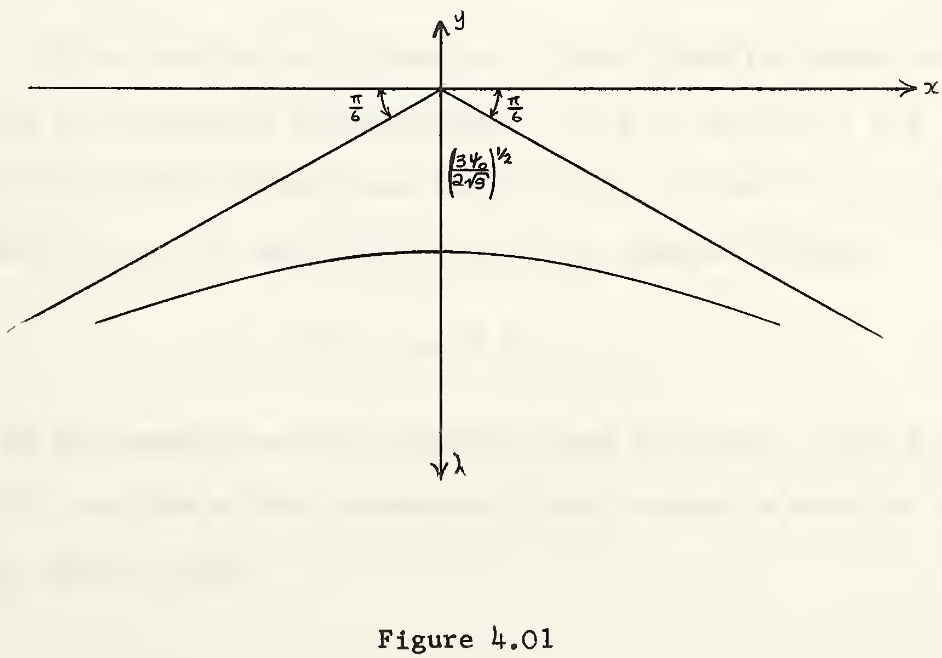


Figure 4.01

CHAPTER FIVE

EXAMPLES

The aim here is to construct some examples of flows which behave like a uniform flow of finite depth at great distances upstream.

We have that the complex velocity is given by equation (4.10):

$$\omega = (U^2 + 2g\lambda)^{\frac{1}{2}} \left[\frac{\sigma(\lambda) - i}{\sigma(\lambda) + i}\right]^{\frac{1}{2}} . \qquad (5.01)$$

Since $\sigma(\lambda)$ is real for real $\lambda \geq -\frac{U^2}{2g}$, then $|\omega| \to U$ as $\lambda \to 0$ through real values. Moreover, since $\sigma(\lambda) = \frac{dx}{dy}$ on the free streamline we must have $\sigma(\lambda) \to -\infty$ far upstream if the streamline is to be horizontal there. Hence, if we arrange it that $\lambda = 0$ corresponds to a point far upstream on the free streamline, then we will have a uniform flow there. In addition, if it can be shown that the total flux of fluid between some particular streamline and the free streamline is finite, then it follows by continuity that the depth of flow far upstream is finite.

If we restrict our attention to those flows for which the free streamline is continually falling, then $\sigma(\lambda) < 0$ for all $\lambda > 0$. In particular we consider flows which have $\sigma(\lambda) \to -\infty$ as $\lambda \to 0$. The streamline is given in the complex z-plane by equation (4.08):

$$z = -i\lambda - \int \sigma(\lambda)\, d\lambda . \qquad (5.02)$$

In most of our examples we will consider flows for which $\int \sigma(\lambda)\, d\lambda \to \infty$ as $\lambda \to 0$. The form of the streamline in the z-plane is shown in the sketch of figure 5.01.

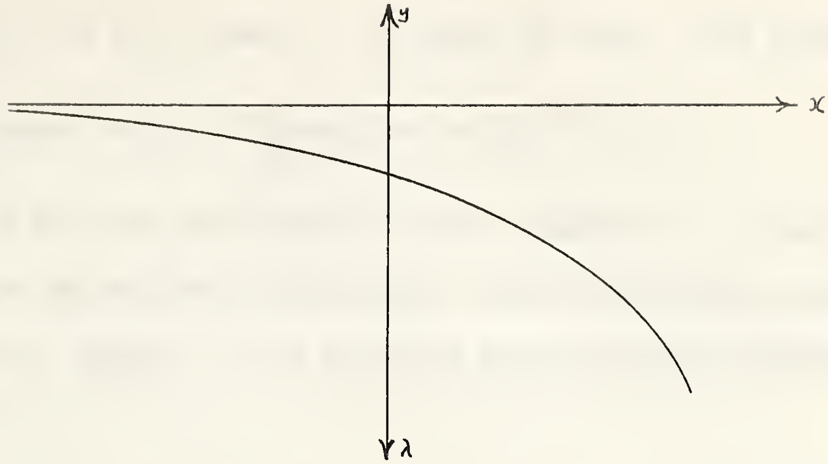


Figure 5.01

It will now be shown that values of $z(\lambda)$ lying beneath the free streamline correspond to $\mathrm{Im}(\lambda) < 0$. We first fix the position of the y-axis along the free streamline by writing equation (5.02) as

$$z(\lambda) = -i\lambda - \int_c^\lambda \sigma(\lambda)\, d\lambda , \tag{5.03}$$

where c is real and positive. Now, if we are given an arbitrary, real, positive number a, then since the free streamline is analytic, $\sigma(\lambda)$ is analytic near $\lambda = a$. Moreover, $\sigma(a) < 0$. Hence,

$$\sigma(\lambda) = \sigma(a) + \sigma'(a)(\lambda-a) + \tfrac{1}{2}\sigma''(a)(\lambda-a)^2 + \ldots$$

for λ sufficiently near a. Equation (5.03) then gives

$$\begin{aligned} z(\lambda) &= -i\lambda - \int_c^\lambda \sigma(\lambda)\, d\lambda \\ &= -i(\lambda-a) - ia - \int_c^a \sigma(\lambda)\, d\lambda - \int_c^\lambda \sigma(\lambda)\, d\lambda \\ &= -ia - \int_c^a \sigma(\lambda)\, d\lambda - i(\lambda-a) - [\sigma(a)(\lambda-a) + \tfrac{1}{2}\sigma'(a)(\lambda-a)^2 + \ldots] \\ &= z(a) - (i + \sigma(a))(\lambda-a) - \tfrac{1}{2}\sigma'(a)(\lambda-a)^2 + \ldots \end{aligned}$$

Now suppose $\lambda = a + i\epsilon$ where ϵ is small and real. This gives

$$z(a+i\epsilon) = z(a) + (1-i\sigma(a))\epsilon + \tfrac{1}{2}\sigma'(a)\epsilon^2 + \ldots \quad .$$

Since $\sigma(a) < 0$, this shows that for small negative ϵ , $z(a+i\epsilon)$ lies below and to the left of the point $z(a)$ on the free streamline. That is to say, $z(a+i\epsilon)$ is in the fluid below the free streamline if $\epsilon = \mathrm{Im}(\lambda) < 0$.

From equation (5.01) it is evident that if $\sigma(\lambda) = -i$, then there will be a singularity in the velocity provided $U^2 + 2g\lambda \neq 0$ at the same point. Define λ_c and z_c by:

$$\left.\begin{aligned} \sigma(\lambda_c) &= -i \ , \\ \text{and} \qquad z_c &= z(\lambda_c) \ . \end{aligned}\right\} \quad (5.04)$$

If we assume that $\sigma(\lambda)$ is analytic near λ_c , then we can determine the nature of the singularity there. Since $\sigma(\lambda_c) = -i$, we can write

$$\sigma(\lambda) = -i + \sigma'(\lambda_c)(\lambda-\lambda_c) + \ldots$$

for values of λ sufficiently near λ_c . We have from equation (5.03) that $\frac{dz}{d\lambda} = -i - \sigma(\lambda)$ and $\frac{d^2z}{d\lambda^2} = -\sigma'(\lambda)$. The first of these gives $\frac{dz}{d\lambda} = 0$ when $\lambda = \lambda_0$. Equation (5.02) implies that $z(\lambda)$ is analytic near $\lambda = \lambda_c$ if $\sigma(\lambda)$ is. Hence

$$z(\lambda) = z_c - \tfrac{1}{2}\sigma'(\lambda_c)(\lambda-\lambda_c)^2 + \ldots$$

near $\lambda = \lambda_c$. We now write

$$\lambda = \lambda_c + \rho e^{i\delta}$$

and consider the situation when $\rho \simeq 0$. The expansions of $z(\lambda)$ and

$\sigma(\lambda)$ become:

$$\sigma = -i + \sigma'(\lambda_c)\rho e^{i\delta} + \dots$$

$$z - z_c = -\tfrac{1}{2}\sigma'(\lambda_c)\rho^2 e^{2i\delta} + \dots$$

when $\rho \simeq 0$. In this notation equation (5.01) becomes

$$\omega = \left[(U^2 + 2g\lambda_c + 2g\rho e^{i\delta})\left(\frac{-2i + \sigma'(\lambda_c)\rho e^{i\delta}}{\sigma'(\lambda_c)\rho e^{i\delta}}\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{-2i}{\rho e^{i\delta}\sigma'(\lambda_c)} \;(U^2 + 2g\lambda_c + 2g\rho e^{i\delta})(1 + \tfrac{i}{2}\sigma'(\lambda_c)\rho e^{i\delta})\right]^{\frac{1}{2}}$$

$$\simeq \sqrt{\frac{2}{\rho}}\; e^{-i\frac{\delta}{2} + i\frac{\pi}{4}} \left[\frac{1}{-\sigma'(\lambda_c)}(U^2 + 2g\lambda_c)\right]^{\frac{1}{2}}$$

where we are keeping only the leading term on the assumption that ρ is small. The case where $U^2 + 2g\lambda_c \doteq 0$ will be dealt with below.

If we now define

$$\left.\begin{aligned} \mu &= \arg[-\sigma'(\lambda_c)] \\ \text{and} \quad \nu &= \arg[U^2 + 2g\lambda_c] \;, \end{aligned}\right\} \quad (5.05)$$

then $\arg\left\{\left[\frac{1}{-\sigma'(\lambda_c)}\;(U^2 + 2g\lambda_c)\right]^{\frac{1}{2}}\right\} = \tfrac{1}{2}(\nu - \mu)$. Hence

$$\arg(\bar{\omega}) = \frac{\delta}{2} - \frac{\pi}{4} - \tfrac{1}{2}(\nu - \mu) \;.$$

Also $\quad \arg(z-z_c) = 2\delta + \mu \;.$

Since $\arg(\bar{\omega})$ is the direction of the fluid velocity, then the streamline through z_c will correspond to values of δ for which the quantities $\arg(\bar{\omega})$ and $\arg(z-z_c)$ are equal or differ by a multiple of π. To find

these, set

$$2\delta + \mu = \frac{\delta}{2} - \frac{\pi}{4} - \tfrac{1}{2}(\nu - \mu) + n\pi .$$

That is

$$\delta = -\frac{1}{3}(\nu + \mu) + \frac{4n-1}{6}\pi .$$

The three independent values of δ correspond to $n = 0, 1, 2$; all others differ from these by a multiple of 2π. We have that

$$\arg(z-z_c) = \frac{1}{3}(\mu - 2\nu) + \left(\frac{4n-1}{3}\right)\pi ,$$

and

$$\arg(\bar{\omega}) = \frac{1}{3}(\mu - 2\nu) + \left(\frac{n-1}{3}\right)\pi .$$

As n increases by two, $\arg(z-z_c)$ increases by $\frac{8\pi}{3}$ and so these two values do not lie on the same physical plane. Hence the streamline entering and leaving the corner must correspond to two consecutive values of n. Since we want $\omega \to \infty$ near the corner, and since we want the flow to be from left to right in figure 5.01, then the pair $n=1$, $n=2$ are inadmissible. We have, then, for $n = 0, 1$,

$n = 0$ $$\arg(z-z_c) = \arg(\bar{\omega}) = \frac{1}{3}(\mu - 2\nu) - \frac{\pi}{3} ;$$

$n = 1$ $$\arg(z-z_c) = \frac{1}{3}(\mu - 2\nu) + \pi , \quad \arg(\bar{\omega}) = \frac{1}{3}(\mu - 2\nu) .$$

The first corresponds to a streamline leaving z_c; the second to one entering. If we define the angle c as

$$c = \frac{\pi}{3} - \frac{1}{3}(\mu - 2\nu) , \tag{5.06}$$

then the situation in the neighborhood of z_c is as appears in figure 5.02.

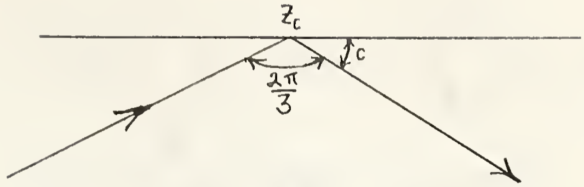


Figure 5.02

The streamline enters the singularity and is rotated through an angle of $-\frac{\pi}{3}$ before it leaves. Since we cannot abide a singularity like this inside the fluid, it is necessary to choose the streamline through z_c or one above it as the bed of our flow. If we choose the one through z_c the situation is that of a flow over a sharp corner of angle $\frac{2\pi}{3}$. In the examples that follow we will refer to this streamline as the bed and to the point z_c as the corner.

Corners with angles other than $\frac{2\pi}{3}$ are also possible. It is the behaviour of $\sigma(\lambda)$ near $\lambda = \lambda_c$ that affects the situation. If, in fact, we suppose that near $\lambda = \lambda_c$, $\sigma(\lambda)$ behaves like

$$\sigma(\lambda) = -i + a(\lambda-\lambda_c)^{\gamma} , \tag{5.07}$$

then other angles are obtained. Here a and γ are constants and $\gamma > 0$. We have in general from equation (5.03) that

$$\begin{aligned} z(\lambda) &= -i\lambda - \int_c^{\lambda_c} \sigma(\lambda)\, d\lambda \\ &= -i\lambda_c - \int_c^{\lambda_c} \sigma(\lambda)\, d\lambda - i(\lambda-\lambda_c) - \int_{\lambda_c}^{\lambda} \sigma(\lambda)\, d\lambda . \end{aligned}$$

If we use equations (5.04) and (5.07) we can write this as

$$z(\lambda) = z_c - i(\lambda-\lambda_c) - \int_{\lambda_c}^{\lambda} [-i - a(\lambda-\lambda_c)^{\gamma}] \, d\lambda$$

$$= z_c - \frac{a}{\gamma + 1} (\lambda-\lambda_c)^{\gamma+1} .$$

Now, as above, if we write

$$\lambda = \lambda_c + \rho e^{i\delta} ,$$

then we obtain

$$\sigma(\lambda) = -i + a\rho^{\gamma} e^{i\gamma\delta} ,$$

and
$$z(\lambda) = z_c - \frac{a}{\gamma + 1} \rho^{\gamma+1} e^{i(\gamma+1)\delta} ,$$

when $\rho \sim 0$. These relations together with equation (5.01) yield

$$\omega = \left[(U^2 + 2g\lambda_c + 2g\rho e^{i\delta}) \left(\frac{-2i + a\rho^{\gamma} e^{i\gamma\delta}}{a\rho^{\gamma} e^{i\gamma\delta}} \right) \right]^{\frac{1}{2}} . \quad (5.08)$$

We now have two situations to consider according as the quantity $U^2 + 2g\lambda_c$ is, or is not, equal to zero. The first possibility can only occur if λ_c is a negative real number and, at the same time, the velocity U far upstream has an appropriate value. One of the examples below exhibits this property. We will discuss both possibilities.

First assume that $U^2 + 2g\lambda_c \neq 0$. Then equation (5.08) gives

$$\omega = \left[\frac{(U^2 + 2g\lambda_c)(-2i)}{a\rho^{\gamma} e^{i\gamma\delta}} \right]^{\frac{1}{2}}$$

where only the largest term has been retained. If we now define

$$\mu_1 = \arg(-a) \quad (5.09)$$

and keep $\nu = \arg(U^2 + 2g\lambda_c)$ as in equation (5.05), then

$$\arg(\bar{\omega}) = \frac{\gamma\delta}{2} - \frac{\pi}{4} - (\nu - \mu_1) .$$

In addition

$$\arg(z-z_c) = (\gamma+1)\delta + \mu_1 .$$

If we again reason that values of δ for which $\arg(\bar{\omega})$ and $\arg(z-z_c)$ differ by a multiple of π correspond to streamlines through z_c, then these values are given by

$$(\gamma+1)\delta + \mu_1 = \frac{\gamma}{2}\delta - \frac{\pi}{4} - \tfrac{1}{2}(\nu - \mu_1) + n\pi ,$$

$$\frac{\gamma+2}{2}\delta = -\tfrac{1}{2}(\nu + \mu_1) + (\frac{4n-1}{4})\pi ,$$

$$\delta = -\frac{\nu + \mu_1}{\gamma + 2} + \frac{4n - 1}{2(\gamma+2)}\pi .$$

This gives

$$\begin{aligned}\arg(z-z_c) &= -\frac{\gamma+1}{\gamma+2}(\nu + \mu_1) + \frac{\pi(\gamma+1)}{2(\gamma+2)}(4n-1) + \mu_1 \\ &= \frac{1}{\gamma+2}[\mu_1 - (\gamma+1)\nu] + \frac{\pi}{2}\,\frac{\gamma+1}{\gamma+2}(4n-1) .\end{aligned}$$

If n increases by two, then $\arg(z-z_c)$ increases by an amount $(\frac{\gamma+1}{\gamma+2})(4\pi)$ which is greater than 2π for all $\gamma > 0$. Hence $n=0$ and $n=1$ give two values of $\arg(z-z_c)$ which lie in the same physical plane. The first means $\arg(z-z_c) = \arg(\bar{\omega})$ and so corresponds to a streamline leaving z_c; the second corresponds to a streamline entering z_c. These two values of $\arg(z-z_c)$ differ by $(2\pi)(\frac{\gamma+1}{\gamma+2})$ with the streamline leaving the corner corresponding to the smaller value. Hence the angle of the corner itself is

$$c_0 = 2\pi - 2\pi(\frac{\gamma+1}{\gamma+2}) = \frac{2\pi}{\gamma+2} . \qquad (5.10)$$

This varies from π down to zero as γ runs from 0 to ∞. The situation is as is shown in figure 5.03.

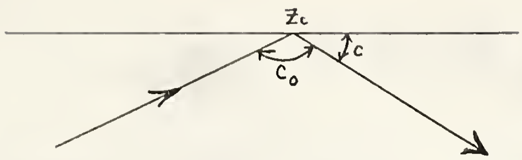


Figure 5.03

The angle c in this case is

$$c = \frac{\pi}{2}\left(\frac{\gamma+1}{\gamma+2}\right) - \frac{1}{\gamma+2}[\mu_1 - (\gamma+1)\nu] \ . \tag{5.11}$$

If $\sigma(\lambda)$ is an analytic function of λ near $\lambda = \lambda_0$, then γ will be an integer. This being the case, then if we suppose that $\sigma(\lambda_c) = -i$, and that the r^{th} derivatives of $\sigma(\lambda)$ is the first one which does not vanish at $\lambda = \lambda_c$, then equation (5.07) implies that $\gamma = r$ and $a = \frac{1}{r!}\sigma^{(r)}(\lambda_c)$. The computation leading to equation (5.06) is the case $\gamma = 1$. Equation (5.11) reduces to equation (5.06) in this case and equation (5.10) gives $c_0 = \frac{2\pi}{3}$. All the examples in this chapter involve this case; that is to say the choices of the function $\sigma(\lambda)$ which we will make will be analytic at $\lambda = \lambda_c$ and will have $\sigma'(\lambda_c) \neq 0$.

A different situation occurs if $U^2 + 2g\lambda_c = 0$. In this case equation (5.08) gives

$$\omega = \left[(2g\rho e^{i\delta})\left(\frac{-2i + a\rho^{\gamma}e^{i\gamma\delta}}{a\rho^{\gamma}e^{i\gamma\delta}}\right)\right]^{\frac{1}{2}} .$$

If $\rho \simeq 0$ this is

$$\omega \sim 2\sqrt{\frac{g}{\rho^{\gamma-1}}} \quad e^{-i(\gamma-1)\delta + i\frac{\pi}{4}} \left[\frac{1}{-a}\right]^{\frac{1}{2}}$$

which, taken with equation (5.09), yields

$$\arg(\bar{\omega}) = (\gamma-1)\delta - \frac{\pi}{4} + \frac{\mu_1}{2} .$$

Again $$\arg(z-z_c) = (\gamma+1)\delta + \mu_1 .$$

As before we set

$$(\gamma+1)\delta + \mu_1 = (\gamma-1)\delta - \frac{\pi}{4} + \frac{\mu_1}{2} + n\pi$$

to obtain values of δ corresponding to streamlines through z_c. This gives

$$\delta = -\frac{\mu_1}{4} + \frac{4n-1}{8}\pi ,$$

and so

$$\arg(z-z_c) = \frac{3-\gamma}{4}\mu_1 + \frac{\gamma+1}{8}(4n-1)\pi$$

gives the orientation of the possible streamlines. Again $n=0$ corresponds to a streamline leaving z_c and $n=1$ corresponds to one entering. The angle c is given by

$$c = \frac{\gamma+1}{8}\pi - \frac{3-\gamma}{4}\mu_1 \tag{5.12}$$

Since $\arg(z-z_c)$ increases by $\frac{\gamma+1}{2}\pi$ as n increases by one, then the angle c_0 of figure 5.03 is given by

$$c_0 = 2\pi - \frac{\gamma+1}{2}\pi = \frac{3-\gamma}{2}\pi . \tag{5.13}$$

This gives that $c_0 \gtrless \pi$ according as $\gamma \lessgtr 1$. If $\gamma > 1$ we have the case of a corner like those which we have been discussing. If

$\gamma = 1$, $c_0 = \pi$ and there is no corner. Finally if $\gamma < 1$ we have $\omega = 0$ at the corner and $c > \pi$. The extreme values are $\gamma = 0$ and $\gamma = 3$. In the first case $c_0 = \frac{3\pi}{2}$ and so the fluid flows inside a corner of angle $\frac{\pi}{2}$. If $\gamma = 3$ then $c_0 = 0$ and so this is the case of flows over a sharp cusp. Values of $\gamma > 3$ give $c_0 < 0$ and so are non physical. It is of some interest to point out that $\gamma = 2$ gives $c_0 = \frac{\pi}{2}$ independently of whether $U^2 + 2g\lambda$ is zero or not.

Since the velocity far upstream is known, then we can calculate the depth there if we know the total flux of fluid. We calculate this by determining the change in the stream function ψ between the bed and the free surface. There are two quite general approaches to this problem. If it happens that the free streamline assumes a slope of -1 at some point then $\sigma = -1$ there and so, since the corner corresponds to $\sigma = -i$, we can find the change in f between these points by integrating its derivative along a portion of the unit circle in the σ-plane. Since, by equation (4.11),

$$f = \int (U^2 + 2g\lambda)^{\frac{1}{2}} (1 + \sigma^2)^{\frac{1}{2}} d\lambda \quad ,$$

then if we denote the change in f between the bed and the free streamline by Δf , we get

$$\Delta f = \int_{-i}^{-1} (U^2 + 2g\lambda)^{\frac{1}{2}} (1 + \sigma^2)^{\frac{1}{2}} \frac{d\lambda}{d\sigma} d\sigma$$

wherever we can invert $\sigma = \sigma(\lambda)$ to obtain λ in terms of σ . If we write

$$\sigma = -e^{i\theta}$$

then $$\sqrt{1 + \sigma^2} = \sqrt{1 + e^{2i\theta}} = e^{i\frac{\theta}{2}} \sqrt{2 \cos \theta} \ .$$

Hence

$$\Delta f = -\sqrt{2} \int_0^{\frac{\pi}{2}} (U^2 + 2g\lambda)^{\frac{1}{2}} (\cos\theta)^{\frac{1}{2}} e^{i\frac{\theta}{2}} \frac{d\lambda}{d\theta} \, d\theta. \tag{5.14}$$

The change in the stream function, $\Delta\psi$, will be the imaginary part of this. Equation (5.14) shows in addition that a sufficient condition that the stream have a finite depth far upstream is that $\left|(U^2+2g\lambda)^{\frac{1}{2}} \frac{d\lambda}{d\theta}\right|$ be bounded for $0 \leq \theta \leq \frac{\pi}{2}$ where, of course, $\lambda = \lambda(\sigma) = \lambda(-e^{i\theta})$.

We can also compute Δf from an integral in the λ-plane in many cases of interest. If the corner corresponds to $\lambda = \lambda_c$ where

$$\lambda_c = R_c e^{-i\alpha_c} ,$$

then, for a wide class of flows, $\lambda = R_c$ will correspond to some point on the free streamline. Hence

$$\Delta f = \int_{R_c e^{-i\alpha_c}}^{R_c} (U^2 + 2g\lambda)^{\frac{1}{2}} (1 + \sigma^2)^{\frac{1}{2}} \, d\lambda .$$

If we set

$$\lambda = R_c e^{-i\theta}$$

this becomes, after a little simplification,

$$\Delta f = iR_c \int_0^{\alpha_c} (U^2 + 2gR_c e^{-i\theta})^{\frac{1}{2}} \sqrt{1 + \sigma^2} \; e^{-i\theta} \, d\theta . \tag{5.15}$$

Here $\sigma = \sigma(\lambda) = \sigma[R_c e^{-i\theta}]$. This method does not require that the function $\sigma(\lambda)$ be inverted and it has the additional advantage that it will work independently of whether σ assumes the value -1 on the free streamline or not. It will fail if the free streamline does not drop to a level where $\lambda = R_c$.

Once the quantity Δf has been obtained the depth of the fluid far upstream can be obtained. The imaginary part of Δf which we will call $\Delta\psi$ represents the total flux of fluid down the stream. Since the velocity upstream is equal to U, the depth h is given by

$$h = \frac{\Delta\psi}{U} . \tag{5.16}$$

In all the examples that follow we will use dimensionless coordinates obtained by including a real positive scale factor b with the dimension of length. In addition the dimensionless quantity $\frac{2gb}{U^2}$ appears quite often and we denote this for simplicity by

$$F = \frac{2gb}{U^2} , \tag{5.17}$$

The position of the corner depends only on $\sigma(\lambda)$, but the orientation of the bed at the corner and the depth far upstream depend on F.

The depth of fluid far downstream can be found from Bernoulli's equation if some assumptions are made. This equation in the form (4.02) gives the speed V on the free streamline as

$$V = [U^2 + 2g\lambda]^{\frac{1}{2}} .$$

If we assume that the velocity of the fluid is nearly uniform, then if we denote by d the thickness of the fluid perpendicular to the free streamline far downstream, we have

$$\Delta\psi = dV .$$

Hence, using equations (5.16) and (5.17)

$$d = \frac{h}{[1 + F\frac{\lambda}{b}]^{\frac{1}{2}}} \tag{5.18}$$

It is assumed here that λ is large enough so that our assumptions are valid.

<u>Example 1 $\sigma(\lambda) = -b^\alpha/\lambda^\alpha \quad \alpha > 1$</u> .

This choice of $\sigma(\lambda)$ ensures that $\sigma \to -\infty$ as $\lambda \to 0$ through positive values. We have

$$-\int_c^\lambda \sigma(\lambda)\, d\lambda = \left.\frac{-b^\alpha}{(\alpha-1)\lambda^{\alpha-1}}\right|_c^\lambda$$

and so the choice of $\alpha > 1$ ensures that this approaches $-\infty$ as $\lambda \to 0$. For convenience choose $c = \infty$. Then we have from equation (5.02) that

$$z = -i\lambda - \frac{b^\alpha}{(\alpha-1)\,\lambda^{\alpha-1}} \quad . \tag{5.19}$$

The free streamline takes the form

$$x = \frac{-b^\alpha}{(\alpha-1)\,\lambda^{\alpha-1}} \tag{5.20}$$

This curve is shown in figure 5.04.

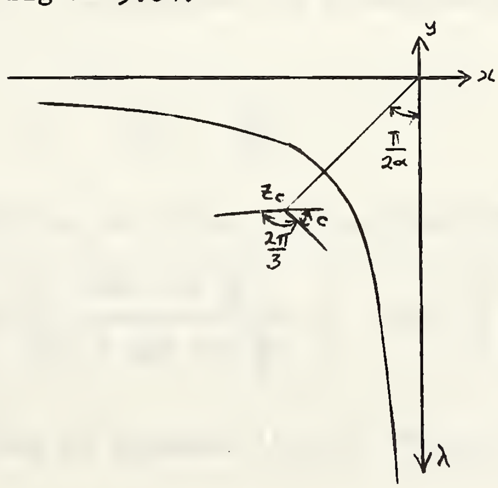


Figure 5.04

In this case λ_c is determined by

$$\sigma(\lambda_c) = -\frac{b^\alpha}{\lambda_c^\alpha} = -i \, .$$

That is

$$\lambda_c = b\, e^{-i\frac{\pi}{2\alpha}} \, . \tag{5.21}$$

This has a negative imaginary part and thus corresponds to a value of z beneath the free streamline. In fact, if we use equation (5.19) we have

$$z_c = -i\lambda_c - \frac{b^\alpha}{\alpha-1}\,\frac{\lambda_c}{\lambda_c^{\,\alpha}}$$

$$= \frac{\alpha b}{\alpha-1}\, e^{-\frac{i\pi}{2}(1+1/\alpha)}$$

as the location of the corner.

We have that $\sigma'(\lambda) = \frac{\alpha b^\alpha}{\lambda^{\alpha+1}}$. This gives

$$\sigma'(\lambda_c) = \frac{\alpha b^\alpha}{\lambda_c^{\,\alpha}\,\lambda_c} = \frac{-\alpha}{b}\, e^{-\frac{\pi i}{2}(1-1/\alpha)} \, .$$

Since this is not zero, and since equation (5.21) shows that λ_c is not a negative real number, then the angle c_0 of figure 5.03 is $\frac{2\pi}{3}$ as is shown in figure 5.04. In addition, if we use one of equations (5.05) then

$$\mu = -\frac{\pi}{2} - \frac{\pi}{2\alpha} \, .$$

Using equation (5.21) in the other of equations (5.05) we have

$$\nu = \tan^{-1}\left[\frac{-2gb\sin\frac{\pi}{2\alpha}}{U^2 + 2gb\cos\frac{\pi}{2}\alpha}\right] = \tan^{-1}\left[\frac{-F\sin\frac{\pi}{2\alpha}}{1 + F\cos\frac{\pi}{2\alpha}}\right]$$

where we have made use of equation (5.17). These results and equation (5.06) give

$$c = \frac{\pi}{3} - \frac{1}{3}\left[-\frac{\pi}{2} + \frac{\pi}{2\alpha} - 2\tan^{-1}\left[\frac{-F\sin\frac{\pi}{2\alpha}}{1 + F\cos\frac{\pi}{2\alpha}}\right]\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{6\alpha} - \frac{2}{3}\tan^{-1}\left[\frac{F\sin\frac{\pi}{2\alpha}}{1 + F\cos\frac{\pi}{2\alpha}}\right] . \qquad (5.22)$$

This relates the angle c and F when α is given.

The quantity ν of equation (5.05) can be written

$$\nu = \arg[1 + F\frac{\lambda_c}{b}]$$

and this is zero when F is zero. Hence we choose the branch of the inverse tangent in equation (5.22) to be the one which is zero when $F = 0$. If α is fixed, then c can easily be shown to be a monotonic decreasing function of F. A sketch of c versus F is shown in Figure 5.05.

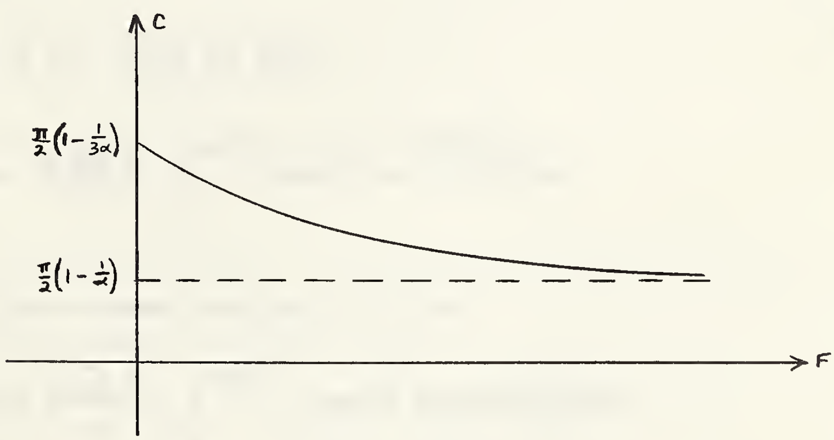


Figure 5.05

This shows that it is impossible to ever arrange it so that $c = 0$, that is that the bed leaves the corner horizontally. The bed will enter the corner horizontally if $c = \frac{\pi}{3}$. Since F is positive this can be arranged only if $1 < \alpha < 3$.

We now examine the flux of fluid down the stream. Since the free streamline has a slope of -1 at some point for all $\alpha > 1$, then we can use the integral (5.14) to compute Δf. We have $\lambda^\alpha = -b^\alpha/\sigma$, and so

$$\lambda = b(-\sigma)^{-1/\alpha} \quad .$$

If $-\sigma = e^{i\theta}$ then $\lambda = b\, e^{-i\theta/\alpha}$, $\dfrac{d\lambda}{d\theta} = -\dfrac{ib}{\alpha}\, e^{-i\theta/\alpha}$, and we get

$$\Delta f = -\sqrt{2}\int_0^{\pi/2} (U^2+2gb\, e^{-i\theta/\alpha})^{\frac{1}{2}} (\cos\theta)^{\frac{1}{2}} e^{i\theta/2}\left(\frac{-ib}{2}\right)e^{-i\theta/\alpha}\, d\theta$$

$$= \frac{ib\sqrt{2}\,U}{\alpha}\int_0^{\pi/2} (1+F\, e^{-i\theta/\alpha})^{\frac{1}{2}}(\cos\theta)^{\frac{1}{2}}\, e^{i\theta(\frac{1}{2}-\frac{1}{\alpha})}\, d\theta \;. \quad (5.23)$$

Two special cases suggest themselves: the case where F is zero and that where it is very large. These correspond respectively to rapid and slow flow upstream.

If $F = 0$ then we have

$$\Delta f = \frac{ib\sqrt{2}\,U}{\alpha}\int_0^{\pi/2} (\cos\theta)^{\frac{1}{2}}\, e^{i\theta(\frac{1}{2}-\frac{1}{\alpha})}\, d\theta \quad .$$

The change in the stream function is, then

$$\Delta\psi = \frac{b\sqrt{2}\,U}{\alpha}\int_0^{\pi/2} (\cos\theta)^{\frac{1}{2}} \cos[\theta(\tfrac{1}{2}-\tfrac{1}{\alpha})]\, d\theta \quad .$$

If on the other hand the flow upstream is very slow then F is large and so, approximately

$$\Delta f = \frac{ib\sqrt{2}\,U\sqrt{F}}{\alpha}\int_0^{\pi/2} (\cos\theta)^{\frac{1}{2}}\, e^{i\frac{\theta}{2}[1-3/\alpha]}\, d\theta \quad .$$

This gives

$$\Delta\psi = \frac{b\sqrt{2}\,U\sqrt{F}}{\alpha}\int_0^{\pi/2}(\cos\theta)^{\frac{1}{2}}\cos[\tfrac{\theta}{2}(1-3/\alpha)]\,d\theta \quad .$$

The other case where the integral for Δf separates easily into real and imaginary parts is when $F = 1$. If this is the case then

$$(1 + F\,e^{-\frac{i\theta}{\alpha}})^{\frac{1}{2}} = [e^{-\frac{i\theta}{2\alpha}}(e^{\frac{i\theta}{2\alpha}} + e^{-\frac{i\theta}{2\alpha}})]^{\frac{1}{2}} = e^{-\frac{i\theta}{4\alpha}}(2\cos\frac{\theta}{2\alpha})^{\frac{1}{2}} \quad .$$

Hence

$$\Delta f = \frac{ib\sqrt{2}\,U}{\alpha}\int_0^{\pi/2}(2\cos\frac{\theta}{2\alpha}\cos\theta)^{\frac{1}{2}}\,e^{i\theta(\frac{1}{2}-\frac{5}{4\alpha})}\,d\theta \quad ,$$

and so

$$\Delta\psi = \frac{2bU}{\alpha}\int_0^{\pi/2}(\cos\frac{\theta}{2\alpha}\cos\theta)^{\frac{1}{2}}\cos[\theta(\tfrac{1}{2}-\tfrac{5}{4\alpha})]d\theta \quad .$$

The values of these integrals enable us to calculate the depth h of the stream upstream and the thickness d perpendicular to the free streamline at a distance λ below the origin. These are given by the relations

$$h = \frac{\Delta\psi}{U} \quad \text{and} \quad d = \frac{h}{(1+F\,\frac{\lambda}{b})^{\frac{1}{2}}} \quad .$$

The results are tabulated below together with the depth of the corner below the origin which is given by $\frac{\alpha b}{\alpha-1}\cos\frac{\pi}{2\alpha}$. Values of $\alpha = 2$ and $\alpha = 3$ were used.

	$F = 0$		$F = 1$		$F = \infty$		Depth
	h	d	h	d	h	d	of Corner
$\alpha=2$	0.85b	0.85b	1.18b	$\frac{1.18b}{(1+\frac{\lambda}{b})^{\frac{1}{2}}}$	$0.83b\sqrt{F}$	$0.83b(\frac{b}{\lambda})^{\frac{1}{2}}$	$\sqrt{2}\,b=1.41b$
$\alpha=3$	0.56b	0.56b	0.79b	$\frac{0.79b}{(1+\frac{\lambda}{b})^{\frac{1}{2}}}$	$0.57b\sqrt{F}$	$0.57b(\frac{b}{\lambda})^{\frac{1}{2}}$	$\frac{3\sqrt{3}}{4}\,b=1.30b$

In the special case where $\alpha = 1$ much of the above work is valid. Here, if c is chosen equal to b, then

$$-\int_c^\lambda \sigma(\lambda)\, d\lambda = b\, \ell n\left(\frac{\lambda}{b}\right) \quad ,$$

and so $$z(\lambda) = -i\lambda + b\, \ell n\left(\frac{\lambda}{b}\right) \quad .$$

The free streamline is given by

$$x = b\, \ell n\left(\frac{\lambda}{b}\right)$$

which is sketched in Figure 5.06.

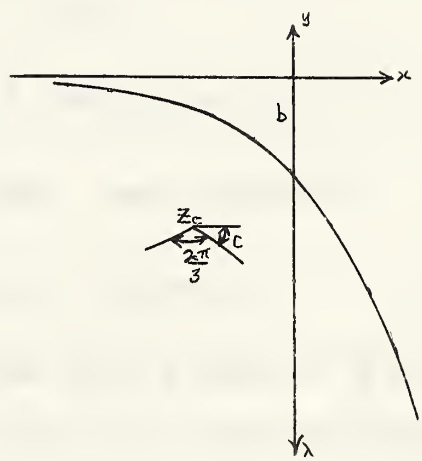


Figure 5.06

It is easily verified that the corner corresponds to $\lambda_c = -ib$ which is given by equation (5.21) with $\alpha = 1$. This in turn gives

$$z_c = -b - i\,\frac{b\pi}{2} \quad .$$

Equation (5.22) is valid when $\alpha = 1$. The result is

$$c = \frac{\pi}{3} - \frac{2}{3}\tan^{-1}(F) \quad .$$

As F increases from zero to infinity, c decreases from $\frac{\pi}{3}$ to zero. These values correspond respectively to the bed entering and leaving the corner horizontally.

The integral (5.23) is valid when $\alpha = 1$ and so the three cases $F = 0$, $F = 1$ and $F = \infty$ correspond to

$$\Delta\psi = b\sqrt{2}\,U \int_0^{\pi/2} [\cos\theta]^{\frac{1}{2}} \cos(\tfrac{\theta}{2})\, d\theta = bU\frac{\pi}{2} ,$$

$$\Delta\psi = 2bU \int_0^{\pi/2} [\cos\tfrac{\theta}{2}\cos\theta]^{\frac{1}{2}} \cos(\tfrac{3\theta}{4})\, d\theta = 1.95\, bU ,$$

$$\Delta\psi = \sqrt{2}\, bU\sqrt{F} \int_0^{\pi/2} [\cos\theta]^{\frac{1}{2}} \cos\theta\, d\theta = 1.24\, bU\sqrt{F} .$$

These give the depth h upstream as

$$h = \frac{\pi b}{2}, \quad 1.95b, \quad \text{and} \quad 1.24b\sqrt{F}$$

when $F = 0$, $F = 1$, and $F = \infty$

respectively. The corner is at a depth of $\frac{\pi}{2}b = 1.57b$ below the origin. Both the cases $F = 1$ and $F = \infty$ have the bed rising to the corner.

The case $F = 0$ is of special interest. It can be considered as a gravity free case and it has the property that the corner and the bed far upstream are both at the same depth. In fact, the whole bed between these points is flat and horizontal at a depth of $\frac{\pi b}{2}$. To see this consider the points corresponding to $\lambda = \frac{-ib}{\xi}$ for $\xi > 1$ and real. This gives that

$$z = -i\left(\frac{-ib}{\xi}\right) + b\,\ell n\left(\frac{-i}{\xi}\right)$$

$$= -b\left[\frac{1}{\xi} + \ell n\,\xi\right] - \frac{ib\pi}{2} ,$$

which traces out the required horizontal curve as ξ runs from 1 to ∞. To see that this is a streamline, we remember that, if $F = \frac{2gb}{U^2} = 0$, then, from equation (4.11)

$$df = U(1+\sigma^2)^{\frac{1}{2}}\, d\lambda \quad .$$

We have $\sigma = -\frac{b}{\lambda} = -i\,\xi$ and so

$$df = U(1 - \xi^2)^{\frac{1}{2}} \left(\frac{ib}{\xi^2}\right) d\xi$$

which is real if $\xi > 1$. Hence the change in f with $\xi > 1$ is real and so this is a streamline. The fact that it is horizontal is further born out by using equation (5.01) with $F = 0$:

$$\omega = U\left(\frac{\sigma-i}{\sigma+i}\right)^{\frac{1}{2}} \quad .$$

This gives $\omega = U\left(\frac{-\xi-1}{-\xi+1}\right)^{\frac{1}{2}} = u\left(\frac{\xi+1}{\xi-1}\right)^{\frac{1}{2}}$ which is real for $\xi > 1$.

This means the velocity is horizontal for all $\xi > 1$.

The thickness d of the stream at a depth λ below the origin is given by equation (5.18). This is measured at right angles to the free streamline and so is nearly horizontal for large enough λ. The values are

$$d = h = \frac{\pi b}{2}\ , \qquad d = \frac{1.95b}{\left(1+\frac{\lambda}{b}\right)^{\frac{1}{2}}} \qquad \text{and} \qquad d = 1.24b\left(\frac{b}{\lambda}\right)^{\frac{1}{2}}$$

if $\quad F = 0, \qquad F = 1, \qquad$ and $\qquad F = \infty$

respectively.

Some sketches appear in figure 5.14 at the end of this chapter which show how the angle c varies with F. The sketches include the three cases $\alpha = 1, 2, 3$. The depth far upstream is indicated on each diagram except where it is infinite. These sketches appear on page 75.

Example 2. $\sigma(\lambda) = -\cot\beta - b/\lambda \quad 0 < \beta < \pi/2$.

Here again $\sigma(\lambda)$ tends to $-\infty$ as $\lambda \to 0$ as it should. We have

$$-\int_c^\lambda \sigma(\lambda)d\lambda = (\lambda\cot\beta + b\ \ell n\ \lambda)\Big|_c^\lambda ,$$

and so if we choose $c = b$ then this and equation (5.02) yield

$$z = -i\lambda + (\lambda-b)\cot\beta + b\ \ell n(\frac{\lambda}{b}) \quad . \qquad (5.24)$$

The free streamline is plotted in Figure 5.07 and is given analytically by

$$x = (\lambda-b)\ \cot\beta + b\ \ell n(\frac{\lambda}{b}) \quad . \qquad (5.25)$$

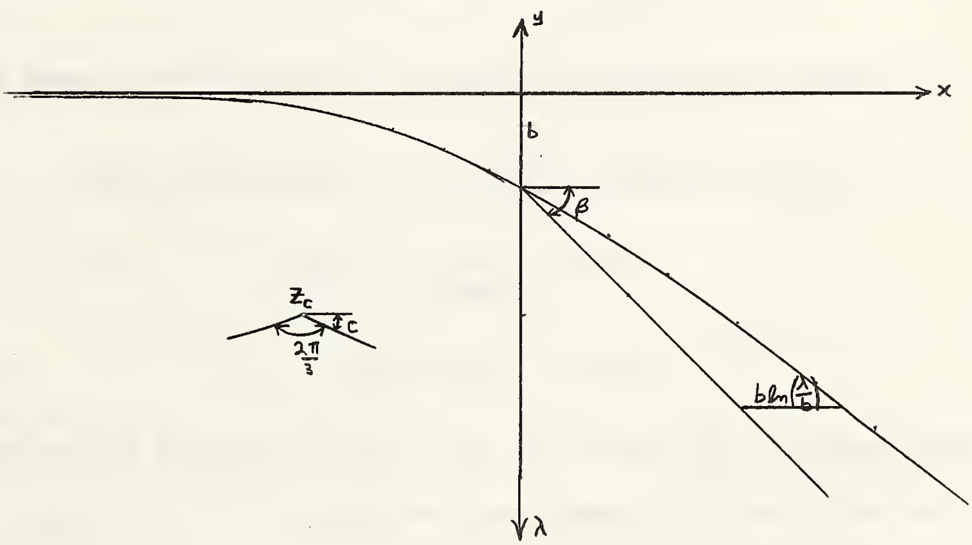


Figure 5.07

The slope of the free streamline tends to $-\tan\beta$ far downstream, but the curve is to the right of the line $x = (\lambda-b)\cot\beta$ by an amount $b\ \ell n(\frac{\lambda}{b})$. The approach to the x axis is exponential here whereas it was algebraic in Example 1.

The corner here is determined by λ_c where

$$\sigma(\lambda_c) = -\cot\beta - \frac{b}{\lambda_c} = -i \quad .$$

Hence
$$\frac{b}{\lambda_c} = -\frac{e^{-i\beta}}{\sin\beta} \ ,$$

and so
$$\lambda_c = -b \sin\beta \, e^{i\beta} = b \sin\beta \, e^{-i(\pi-\beta)} \tag{5.26}$$

as is consistent with the result that $\text{Im}(\lambda_c) < 0$. This of course ensures that $z_c = z(\lambda_c)$ lies beneath the free streamline. Since $\lambda_c(\cot\beta - i) = -b$, then

$$z_c = \lambda_c(\cot\beta - i) - b\cot\beta + b\,\ell n(\frac{\lambda_c}{b})$$

$$= -b(1+\cot\beta) + b\,\ell n \sin\beta + i\,b(\beta-\pi)$$

$$= -b[1+\cot\beta+\ell n\cot\beta] - i\,b(\pi-\beta) \ .$$

This gives the position of the corner in the third quadrant.

For this example $\sigma'(\lambda) = \frac{b}{\lambda^2}$. Hence we have

$$\sigma'(\lambda_c) = \frac{1}{b\sin^2\beta} \, e^{-2i\beta}$$

and so
$$\mu = \arg[-\sigma'(\lambda_c)] = \pi - 2\beta$$

using one of equations (5.05). We again have the situation where $\sigma'(\lambda_c) \neq 0$ and λ_c is not a negative real number. We have also that

$$\nu = \arg(U^2 + 2g\lambda_c)$$

$$= \arg(U^2 - 2gb\sin\beta \, e^{i\beta})$$

$$= \tan^{-1}[\frac{-F\sin^2\beta}{1-F\sin\beta\cos\beta}]$$

where $F = \frac{2gb}{U^2}$ as usual. Finally equation (5.06) gives

$$c = \frac{\pi}{3} - \frac{1}{3}[\pi - 2\beta - 2\tan^{-1}[\frac{-F\sin^2\beta}{1-F\sin\beta\cos\beta}]]$$

$$= \frac{2\beta}{3} - \frac{2}{3}\tan^{-1}[\frac{F\sin^2\beta}{1-F\sin\beta\cos\beta}] \ . \tag{5.27}$$

Again c is a monotonic decreasing function of F and takes the value $\frac{2\beta}{3}$ when $F = 0$. Here the curve is concave down if $F < \cot\beta$ and concave up if $F > \cot\beta$. A sketch of c versus F is found in Figure 5.08.

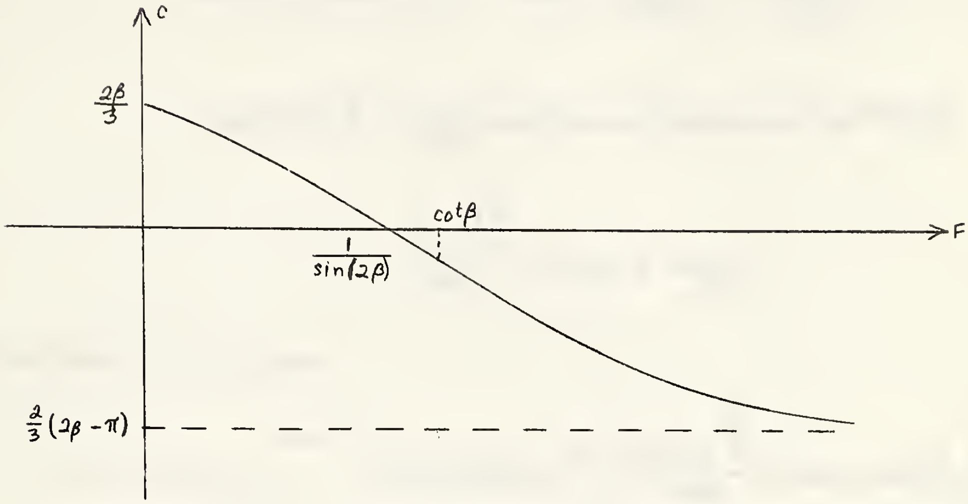


Figure 5.08

The inflection point is to the left (right) of $F = \frac{1}{\sin(2\beta)}$ when β is greater than (less than) $\frac{\pi}{4}$.

The bed will leave the corner horizontally for $F = \csc(2\beta)$ and this is possible for all β in the range $0 < \beta < \frac{\pi}{2}$. However, since $\frac{2\beta}{3} < \frac{\pi}{3}$, it is impossible to arrange it so that the bed enters the corner horizontally if $0 < \beta < \frac{\pi}{2}$. If, in fact, $\beta = \frac{\pi}{2}$ we have the flow discussed in Example 1 with $\alpha = 1$. This has the bed entering horizontally when $F = 0$.

In this example, unless $\beta > \frac{\pi}{4}$ σ will never be equal to -1 on the free streamline, and so we cannot use the integral (5.14) to compute Δf. However (5.15) will work for all β. We have that

$$\lambda_c = b \sin\beta \, e^{-i(\pi-\beta)} = R_c \, e^{-i\alpha_c} ,$$

and so $R_c = b \sin\beta$ and $\alpha_c = \pi-\beta$. If we set

$$\lambda = R_c e^{-i\theta} = b \sin\beta \, e^{-i\theta} ,$$

then

$$\sigma(\lambda) = -\cot\beta - \frac{b}{\lambda} = -\cot\beta - \frac{e^{i\theta}}{\sin\beta} = \frac{-\cos\beta - e^{i\theta}}{\sin\beta} .$$

Hence

$$[1+\sigma^2(\lambda)]^{\frac{1}{2}} = \frac{1}{\sin\beta}[\sin^2\beta+(\cos^2\beta+2\cos\beta e^{i\theta}+e^{2i\theta})]^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}\, e^{i\frac{\theta}{2}}}{\sin\beta} [\cos\beta + \cos\theta]^{\frac{1}{2}} .$$

Finally integral (5.15) gives

$$\Delta f = ib \sin\beta \int_0^{\pi-\beta} (U^2+2gb\sin\beta e^{-i\theta})^{\frac{1}{2}}\left(\frac{\sqrt{2}\, e^{i\frac{\theta}{2}}}{\sin\beta}\right)(\cos\beta+\cos\theta)^{\frac{1}{2}} e^{-i\theta} d\theta$$

$$= ib \sqrt{2}\, U \int_0^{\pi-\beta} (1+F\sin\beta e^{-i\theta})^{\frac{1}{2}}(\cos\beta + \cos\theta)^{\frac{1}{2}} e^{-\frac{i\theta}{2}} d\theta . \quad (5.28)$$

This again separates into real and imaginary parts easily for three choices of F. If U is very large, then F is nearly zero and so, approximately,

$$\Delta f = ib \sqrt{2}\, U \int_0^{\pi-\beta} (\cos\beta+\cos\theta)^{\frac{1}{2}} e^{-i\frac{\theta}{2}} d\theta .$$

This gives

$$\Delta\psi = b \sqrt{2}\, U \int_0^{\pi-\beta} (\cos\beta+\cos\theta)^{\frac{1}{2}} \cos \frac{\theta}{2} d\theta .$$

If, on the other hand, U is very small, then F is large and we have, again approximately, that

$$\Delta f = \sqrt{2}\, ibU(F\sin\beta)^{\frac{1}{2}} \int_0^{\pi-\beta} (\cos\beta+\cos\theta)^{\frac{1}{2}} e^{-i\theta} d\theta .$$

Hence

$$\Delta\psi = bU\sqrt{2}(F\sin\beta)^{\frac{1}{2}} \int_0^{\pi-\beta} (\cos\beta+\cos\theta)^{\frac{1}{2}} \cos\theta \, d\theta \quad .$$

One more special case suggests itself: If we choose $F = \frac{1}{\sin\beta}$ then we can write:

$$(1+F\sin\beta e^{-i\theta})^{\frac{1}{2}} = e^{-\frac{i\theta}{4}} (2\cos\frac{\theta}{2})^{\frac{1}{2}} \quad .$$

Hence equation (5.26) becomes

$$\Delta f = 2ibU \int_0^{\pi-\beta} [\cos\frac{\theta}{2}(\cos\beta+\cos\theta)]^{\frac{1}{2}} e^{-\frac{3i\theta}{4}} d\theta \quad .$$

Here then,

$$\Delta\psi = 2bU \int_0^{\pi-\beta} [\cos\frac{\theta}{2}(\cos\beta+\cos\theta)]^{\frac{1}{2}} \cos\frac{3\theta}{4} \, d\theta$$

when $F = \frac{1}{\sin\beta}$.

As in Example 1, these integrals were evaluated for special values of β. The values chosen were $\beta = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$. These results were used to compute the depth h far upstream and the thickness d at great distances downstream. The corner in this case is at a depth $(\pi-\beta)b$ below the origin. All these results appear in tabulated form below.

	$F = 0$		$F = 1/\sin\beta$		$F = \infty$		Depth of Corner
	h	d	h	d	h	d	
$\beta = \frac{\pi}{6}$	2.93b	2.93b	3.06b	$\frac{3.06b}{[1+\frac{2\lambda}{6}]^{\frac{1}{2}}}$	$1.05b\sqrt{F}$	$1.05b(\frac{b}{\lambda})^{\frac{1}{2}}$	$\frac{5\pi}{6}b=2.62b$
$\beta = \frac{\pi}{4}$	2.69b	2.69b	2.92b	$\frac{2.92b}{[1+\frac{\sqrt{2}\lambda}{b}]^{\frac{1}{2}}}$	$1.30b\sqrt{F}$	$1.30b(\frac{b}{\lambda})^{\frac{1}{2}}$	$\frac{3\pi}{4}b=2.36b$
$\beta = \frac{\pi}{3}$	2.36b	2.36b	2.68b	$\frac{2.68b}{[1+\frac{2\lambda}{\sqrt{3}b}]^{\frac{1}{2}}}$	$1.42b\sqrt{F}$	$1.42b(\frac{b}{\lambda})^{\frac{1}{2}}$	$\frac{2\pi}{3}b=2.09b$

Sketches of the case $\beta = \frac{\pi}{4}$ appear in Figure 5.15. They give an indication of how the angle c varies with changes in F. Where h is finite, it is indicated on the diagram. These sketches appear on page 76.

Example 3. $\sigma(\lambda) = -\cot\beta[1 + \frac{b^2}{\lambda^2}]$. $0 < \beta < \pi/2$.

In this case we have that

$$-\int_c^\lambda \sigma(\lambda)\, d\lambda = \cot\beta\ [\lambda - \frac{b^2}{\lambda}]_c^\lambda = \cot\beta\ [\lambda - \frac{b^2}{\lambda}]$$

if we choose $c = b$. Hence, using equation (5.02), we have

$$z = -i\lambda + \cot\beta[\lambda - \frac{b^2}{\lambda}] \quad . \tag{5.29}$$

The free streamline is given by

$$x = \cot\beta\ [\lambda - \frac{b^2}{\lambda}] , \tag{5.30}$$

and is sketched in Figure 5.09. If $\lambda \simeq 0$, then

$$x \simeq \frac{-b^2 \cot\beta}{\lambda}$$

while if $\lambda >> 0$, we have

$$x \simeq \lambda \cot \beta \quad .$$

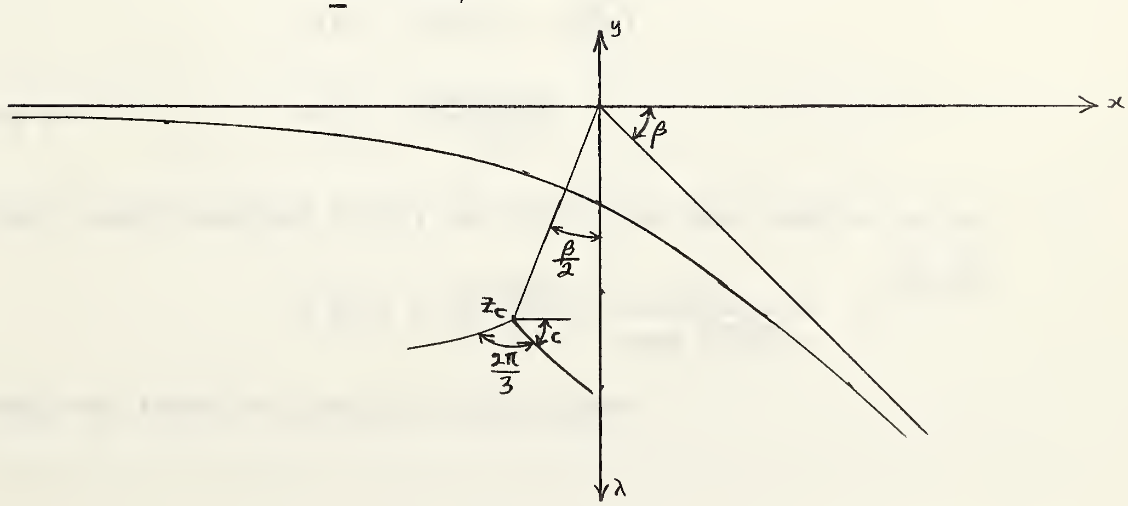


Figure 5.09

Hence the curve approaches the x-axis and the line $x = \lambda \cot\beta$ algebraically. The slope of the free streamline at the point $(x,y) = (0,-b)$ is $-\frac{1}{2}\tan\beta$.

We have λ_c given by

$$\sigma(\lambda_c) = -\cot\beta\ [1+\frac{b^2}{\lambda^2}] = -i \quad .$$

This gives $\frac{b^2}{\lambda_c^2} = i\tan\beta - 1 = \frac{-1}{\cos\beta} e^{-i\beta}$. Hence

$$\lambda_c^2 = -b^2 \cos\beta\ e^{i\beta} \ . \tag{5.31}$$

Choosing $-1 = e^{-i\pi}$ so as to make $\mathrm{Im}(\lambda_c) < 0$, this gives

$$\lambda_c = b\sqrt{\cos\beta}\ \ e^{-i(\frac{\pi}{2}-\frac{\beta}{2})} \ . \tag{5.32}$$

Taken with equation (5.29) this gives, after some algebra, that

$$\begin{aligned} z_c &= \lambda_c(\cot\beta - i) - b^2\cot\beta\ (\frac{1}{\lambda_c}) \\ &= \frac{2b\sqrt{\cos\beta}}{\sin\beta}\ e^{-i(\frac{\pi}{2}+\frac{\beta}{2})} \end{aligned}$$

is the location of the corner.

We now calculate the angle c. We have that

$$\sigma(\lambda) = -\cot\beta\ [1 + \frac{b^2}{\lambda^2}]$$

and so

$$\sigma'(\lambda) = \frac{2b^2\cot\beta}{\lambda^3} \ .$$

Hence using equations (5.31) and (5.32) and some algebra we get

$$\sigma'(\lambda_c) = \frac{2b^2\cot\beta}{\lambda_c\ \lambda_c^2} = \frac{-2}{\sin\beta\sqrt{\cos\beta}\ b}\ e^{i(\frac{\pi}{2}-\frac{3\beta}{2})} \ .$$

Hence the first of equations (5.05) gives

$$\mu = \arg(-\sigma'(\lambda_c)) = \frac{\pi}{2} - \frac{3\beta}{2} \ .$$

The second of equations (5.05) gives

$$\nu = \arg(U^2 + 2g\lambda_c)$$

$$= \arg\left(U^2 + 2gb\sqrt{\cos\beta}\ e^{-i(\frac{\pi}{2}-\frac{\beta}{2})}\right)$$

$$= \tan^{-1}\left[\frac{-F\sqrt{\cos\beta}\cos\frac{\beta}{2}}{1 + F\sqrt{\cos\beta}\sin\frac{\beta}{2}}\right] ,$$

where $F = \frac{2gb}{U^2}$. These, taken with equation (5.06) gives

$$c = \frac{\pi}{3} - \frac{1}{3}\left[\frac{\pi}{2} - \frac{3\beta}{2} - 2\tan^{-1}\left[\frac{-F\sqrt{\cos\beta}\cos\frac{\beta}{2}}{1 + F\sqrt{\cos\beta}\sin\frac{\beta}{2}}\right]\right]$$

$$= \frac{\pi}{6} + \frac{\beta}{2} - \frac{2}{3}\tan^{-1}\left[\frac{F\sqrt{\cos\beta}\cos\frac{\beta}{2}}{1 + F\sqrt{\cos\beta}\sin\frac{\beta}{2}}\right] . \qquad (5.33)$$

This gives c as a decreasing function of F. The curve is sketched in Figure 5.10.

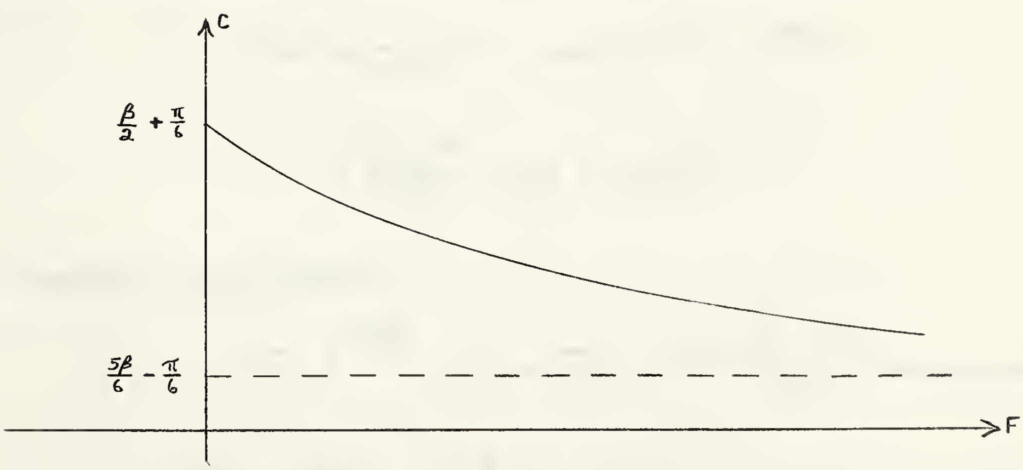


Figure 5.10

The range includes $c = 0$ if $\beta \leq \frac{\pi}{5}$ and so the bed can be made to leave the corner horizontally only if this condition is satisfied. The value

$c = \frac{\pi}{3}$ can occur only if $\beta \geq \frac{\pi}{3}$. Hence this is the condition which must be met in order that we can choose F so that the bed enters the corner horizontally.

Here, as in Example 2, σ need not necessarily equal -1 at some point of the free streamline. However we can again set up an integral for Δf in the λ-plane using equation (5.15). We have that

$$\lambda_c = b\sqrt{\cos\beta}\; e^{-i(\frac{\pi}{2}-\frac{\beta}{2})}$$

and so $R_c = b\sqrt{\cos\beta}$ and $\alpha_c = \frac{\pi}{2} - \frac{\beta}{2}$. If we again write

$$\lambda = R_c\, e^{-i\theta} = b\sqrt{\cos\beta}\; e^{-i\theta}$$

then

$$\sigma^2 = \cot^2\beta\,[1 + \frac{2b^2}{\lambda^2} + \frac{b^4}{\lambda^4}]$$

$$= \frac{1}{\sin^2\beta}\,[\cot^2\beta + 2\cos\beta\, e^{2i\theta} + e^{4i\theta}] \;,$$

using the equation $\sigma(\lambda) = -\cot\beta[1 + \frac{b^2}{\lambda^2}]$. It follows that

$$\sqrt{1+\sigma^2} = \frac{1}{\sin\beta}\,[1 + 2\cos\beta\, e^{2i\theta} + e^{4i\theta}]^{\frac{1}{2}}$$

$$= \frac{\sqrt{2}\, e^{i\theta}}{\sin\beta}\,[\cos\beta + \cos 2\theta]^{\frac{1}{2}} \;.$$

Hence equation (5.15) becomes

$$\Delta f = ib\sqrt{\cos\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (U^2+2gb\sqrt{\cos\beta}\; e^{-i\theta})^{\frac{1}{2}}\, \frac{\sqrt{2}\, e^{i\theta}}{\sin\beta}\,(\cos\beta+\cos 2\theta)^{\frac{1}{2}} e^{-i\theta}\, d\theta$$

$$= \frac{i\sqrt{2}\, b\sqrt{\cos\beta}\; U}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (1+F\sqrt{\cos\beta}\; e^{-i\theta})^{\frac{1}{2}}(\cos\beta+\cos 2\theta)^{\frac{1}{2}}\, d\theta \;, \tag{5.34}$$

where $F = \frac{2gb}{U^2}$.

If U is very large, then F is nearly zero and we can write, approximately, that

$$\Delta f = \frac{ib\sqrt{2\cos\beta}\ U}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (\cos\beta+\cos 2\theta)^{\frac{1}{2}}\, d\theta \quad .$$

Hence the change in the value of the stream function from the bed to the free streamline is

$$\Delta\psi = \frac{bU\sqrt{2\cos\beta}}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (\cos\beta+\cos 2\theta)^{\frac{1}{2}}\, d\theta \quad .$$

In the case where U is very small, F is large and

$$\Delta f = \frac{iU\sqrt{2}\ b\ \sqrt{F}(\cos\beta)^{3/4}}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (\cos\beta+\cos 2\theta)^{\frac{1}{2}}\ e^{-\frac{i\theta}{2}}\, d\theta \quad .$$

Hence:
$$\Delta\psi = \frac{U\sqrt{2}\ b\ \sqrt{F}(\cos\beta)^{3/4}}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} (\cos\beta+\cos 2\theta)^{\frac{1}{2}}\cos\frac{\theta}{2}\, d\theta \quad .$$

As in Example 2 there is one other value of F which allows us to split the integral in equation (5.37) into real and imaginary parts. If $F = \dfrac{1}{\sqrt{\cos\beta}}$ then

$$(1+F\sqrt{\cos\beta}\ e^{-i\theta})^{\frac{1}{2}} = e^{-i\frac{\theta}{4}}\ (2\cos\frac{\theta}{2})^{\frac{1}{2}} \quad ,$$

and so

$$\Delta f = \frac{2ib\sqrt{\cos\beta}\ U}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} e^{-i\frac{\theta}{4}}\ [\cos\frac{\theta}{2}(\cos\beta+\cos 2\theta)]^{\frac{1}{2}}\, d\theta \quad .$$

Hence
$$\Delta\psi = \frac{2bU\sqrt{\cos\beta}}{\sin\beta}\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} [\cos\frac{\theta}{2}(\cos\beta+\cos 2\theta)]^{\frac{1}{2}}\ \cos\frac{\theta}{4}\, d\theta \quad ,$$

when $F = \dfrac{1}{\sqrt{\cos\beta}}$.

As in the above, these integrals for $\Delta\psi$ have been evaluated for $\beta = \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$. The depth h upstream and the thickness d downstream perpendicular to the free streamline are tabulated below, together with the depth below the origin of the corner. The latter is given by $\frac{2b\sqrt{\cos\beta}}{\sin\beta} \cos\frac{\beta}{2}$.

	$F = 0$		$F = 1/\sqrt{\cos\beta}$		$F = \infty$		Depth
	h	d	h	d	h	d	of Corner
$\beta = \frac{\pi}{6}$	3.31b	3.31b	4.54b	$\frac{4.54b}{\left[1+\frac{\sqrt{2\lambda}}{\sqrt[4]{3}\, b}\right]^{\frac{1}{2}}}$	$3.04b\sqrt{F}$	$3.04b(\frac{b}{\lambda})^{\frac{1}{2}}$	3.60b
$\beta = \frac{\pi}{4}$	1.88b	1.88b	3.06b	$\frac{3.06b}{\left[1+\frac{\sqrt[4]{2}\sqrt{2\lambda}}{b}\right]^{\frac{1}{2}}}$	$1.65b\sqrt{F}$	$1.65b(\frac{b}{\lambda})^{\frac{1}{2}}$	2.12b
$\beta = \frac{\pi}{3}$	1.10b	1.10b	1.52b	$\frac{1.52b}{\left[1+\frac{\sqrt{2\lambda}}{b}\right]^{\frac{1}{2}}}$	$0.89b\sqrt{F}$	$0.89b(\frac{b}{\lambda})^{\frac{1}{2}}$	1.41b

<u>Example 4 $\sigma(\lambda) = \frac{-\cot\beta}{1-e^{-\lambda/b}}$. $0 < \beta < \pi/2$.</u>

Here again $\sigma(\lambda) \to -\infty$ as $\lambda \to 0$. In this case we have

$$-\int_c^\lambda \sigma(\lambda)d\lambda = \cot\beta \int_c^\lambda \frac{e^{\lambda/b}}{e^{\lambda/b}-1}\, d\lambda = b\cot\beta[\ln(e^{\lambda/b}-1)]_c^\lambda .$$

If we choose $c = b\ln 2$, then $\ln(e^{c/b}-1) = 0$ and so

$$z(\lambda) = -i\lambda + b\cot\beta \ln[e^{\lambda/b}-1] . \tag{5.35}$$

The equation of the free streamline is

$$x = b\cot\beta \ln[e^{\lambda/b}-1] . \tag{5.36}$$

If $\lambda >> 0$ then $x \simeq b \cot\beta \left(\frac{\lambda}{b}\right) = \lambda \cot \beta$. If, on the other hand, $\lambda \simeq 0$, then $x \simeq b \cot \beta \, \ell n \left(\frac{\lambda}{b}\right)$, or $\lambda \simeq b \, e^{\frac{\tan\beta}{b} x}$. This is an exponential function for $x << 0$ and so the approach to the negative x-axis is exponential. In addition, if we write

$$x = b \cot\beta \, \ell n[e^{\lambda/b} (1-e^{-\lambda/b})]$$

$$= \lambda \cot\beta + b \cot\beta \, \ell n(1-e^{-\lambda/b}] \; ,$$

then for large λ, $x - \lambda\cot\beta \simeq -b \cot\beta \, e^{-\lambda/b}$ which shows that the free streamline approaches the line $x = \lambda \cot\beta$ exponentially. The slope of the free streamline is $-\frac{1}{2} \tan\beta$ at the point $(x,y) = (0,-b \, \ell n \, 2)$. The curve is sketched in Figure 5.11.

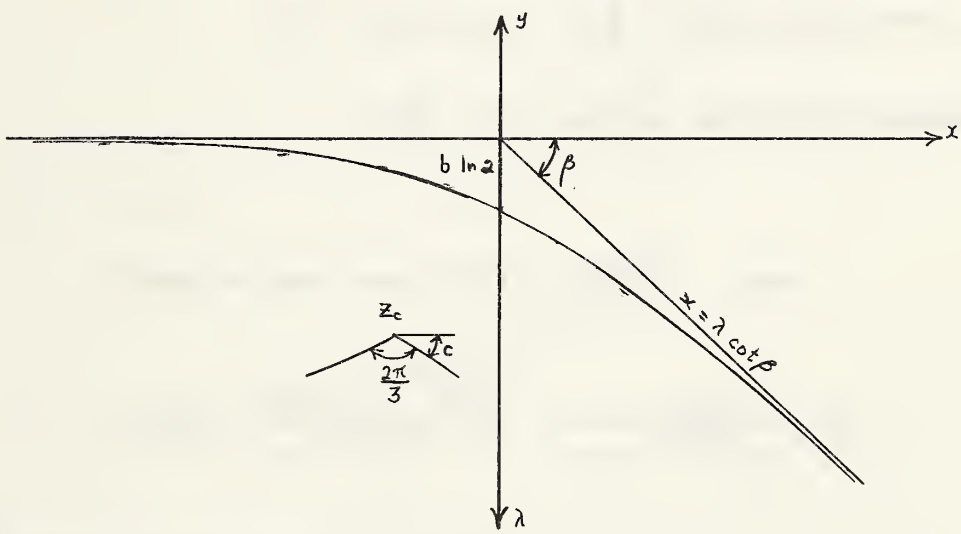


Figure 5.11

In this example, λ_c is given by

$$(\lambda_c) = \frac{-\cot \beta}{1-e^{-\lambda_c/b}} = -i \quad . \tag{5 37}$$

This gives $-i \cot \beta = 1 - e^{-\lambda_c/b}$, and so

$$e^{-\lambda_c/b} = 1 + i \cot \beta = \frac{ie^{-i\beta}}{\sin \beta} \quad .$$

Finally $$\frac{\lambda_c}{b} = \ell n \sin\beta - i(\frac{\pi}{2} - \beta) \quad . \tag{5.38}$$

To compute the position of the corner, z_c, we first calculate $e^{\lambda_c/b} - 1$. Equation (5.37) gives

$$e^{\lambda_c/b} - 1 = -i \cot\beta\, e^{\lambda_c/b} \quad ,$$

and so

$$\ell n[e^{\lambda_c/b} - 1] = \ell n \cot\beta + \lambda_c/b - i\,\frac{\pi}{2}$$

$$= \ell n \cos\beta - i(\pi - \beta)$$

using equation (5.38). Finally, then, equation (5.35) yields

$$\frac{z_c}{b} = -i[\ell n \sin\beta - i(\frac{\pi}{2} - \beta)] + \cot\beta[\ell n \cos\beta - i(\pi - \beta)]$$

$$= -[(\frac{\pi}{2} - \beta) + \cot\beta\, \ell n \sec\beta] - i[\ell n \sin\beta + \cot\beta(\pi - \beta)] . \tag{5.39}$$

Since in this case $\sigma(\lambda) = \dfrac{-\cot\beta}{1-e^{-\lambda/b}}$, then

$$\sigma'(\lambda) = \frac{\cot\beta(-e^{-\lambda/b})(\frac{-1}{b})}{[1-e^{-\lambda/b}]^2} = \frac{-\cot\beta}{b[1-e^{-\lambda/b}][1-e^{\lambda/b}]} \quad .$$

We have from equation (5.37) that

$$[1 - e^{-\lambda_c/b}] = -i \cot\beta$$

and $$[1 - e^{\lambda_c/b}] = i \cot\beta\, e^{\lambda_c/b} \quad ,$$

and so

$$\sigma'(\lambda_c) = \frac{-\cot\beta}{b[-i\cot\beta][i\cot\beta e^{\lambda_c/b}]} = -\frac{\tan\beta}{b}\;\frac{e^{i(\frac{\pi}{2}-\beta)}}{\sin\beta}$$

using equation (5.38). Hence

$$\mu = \arg(-\sigma'(\lambda_c)) = \frac{\pi}{2} - \beta \quad .$$

In addition

$$\begin{aligned}\nu &= \arg[U^2 + 2g\lambda_c] \\ &= \arg[U^2 + 2gb(\ell n \sin\beta - i(\tfrac{\pi}{2}-\beta))] \\ &= \tan^{-1}\left[\frac{-F\ (\frac{\pi}{2} - \beta)}{1 + F\ \ell n \sin\beta}\right] \quad .\end{aligned}$$

where, again, $F = \frac{2gb}{U^2}$. Using equation (5.06) with these results we obtain:

$$\begin{aligned}c &= \frac{\pi}{3} - \frac{1}{3}\left[\frac{\pi}{2} - \beta - 2\tan^{-1}\left[\frac{-F\ (\frac{\pi}{2} - \beta)}{1 + F\ \ell n \sin\beta}\right]\right] \\ &= \frac{\pi}{6} + \frac{\beta}{3} - \frac{2}{3}\tan^{-1}\left[\frac{F(\frac{\pi}{2} - \beta)}{1 + F\ \ell n \sin\beta}\right].\end{aligned}$$

As in previous examples this is a decreasing function of F if β is fixed. The angle c vanishes when

$$F = [(\tfrac{\pi}{2}-\beta)\cot(\tfrac{\pi}{4}+\tfrac{\beta}{2}) + \ell n \csc\beta]^{-1} \quad .$$

Since the right hand side is positive for all values of β in the range $0 < \beta < \frac{\pi}{2}$, then this shows that for each β there exists an F which makes $c = 0$, i.e., which makes the bed leave the corner horizontally.

When $F = 0$, $c = \frac{\pi}{6} + \frac{\beta}{3}$. When $F \to \infty$,

$$\begin{aligned}c &\to \frac{\beta}{3} + \frac{\pi}{6} - \frac{2}{3}\left[\tan^{-1}\left(-\frac{\frac{\pi}{2} - \beta}{\ell n \csc\beta}\right)\right] \\ &= \frac{\beta}{3} + \frac{\pi}{6} - \frac{2}{3}\left[\pi - \tan^{-1}\left(\frac{\frac{\pi}{2} - \beta}{\ell n \csc\beta}\right)\right] \\ &= \frac{\beta}{3} - \frac{\pi}{2} + \frac{2}{3}\tan^{-1}\left(\frac{\frac{\pi}{2} - \beta}{\ell n \csc\beta}\right)\end{aligned}$$

where the inverse tangent in the last expression has its principal value.

Since, for given β, $c \leq \frac{\pi}{6} + \frac{\beta}{3}$, then it is impossible that $c = \frac{\pi}{3}$ unless $\beta = \frac{\pi}{2}$. That is to say, for any $\beta < \frac{\pi}{2}$, it is impossible to choose F so that the bed enters the corner horizontally. If $\beta = \frac{\pi}{2}$, then F must be zero in order that $c = \frac{\pi}{3}$.

Example 5 $\sigma(\lambda) = -\sec^2\left(\frac{\lambda}{b} - \frac{\pi}{2}\right)$.

This example, unlike the others, has the property that the free streamline is bounded in the y-direction. We have $-\int_c^\lambda \sigma(\lambda)d\lambda = \int_c^\lambda \sec^2\left(\frac{\lambda}{b} - \frac{\pi}{2}\right)d\lambda = b\tan\left(\frac{\lambda}{b} - \frac{\pi}{2}\right)$ if c is chosen equal to $\frac{b\pi}{2}$. This gives

$$z = -i\lambda + b\tan\left(\frac{\lambda}{b} - \frac{\pi}{2}\right). \tag{5.40}$$

The free streamline is sketched in Figure 5.12.

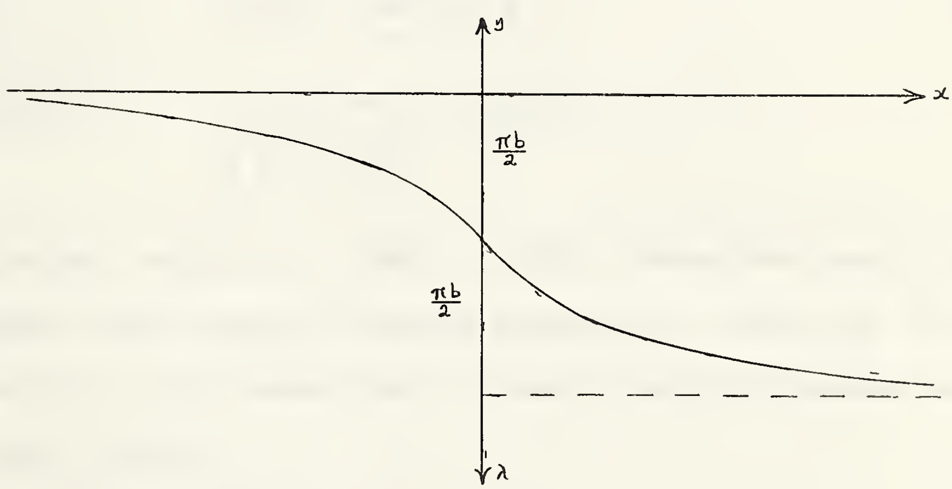


Figure 5.12

The equation of the free streamline is

$$x = b\tan\left(\frac{\lambda}{b} - \frac{\pi}{2}\right). \tag{5.41}$$

The corner of the flow is given by λ_c where

$$\sigma(\lambda_c) = - \sec^2\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = - i \quad ,$$

$$\cos^2\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}} \quad ,$$

$$\cos\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \pm \left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] \quad .$$

If we write $\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = p + iq$, then if we choose $\cos\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ we have

$$\cos p \cosh q - i \sin p \sinh q = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \ .$$

Hence $$\cos p \cosh q = \frac{1}{\sqrt{2}} = \sin p \sinh q \ .$$

This gives $$\frac{1}{2} = \cos^2 p \cosh^2 q = (1-\sin^2 p)(1+\sinh^2 q) \quad .$$

If we use the fact that $\sin^2 p \sinh^2 q = \frac{1}{2}$, this gives

$$\sin^2 p = \sinh^2 q \ ,$$

and so $$\sin^4 p = \sinh^4 q = \frac{1}{2} \quad .$$

These yield $$p = \pm\, 0.999, \quad \pi \pm 0.999$$

$$q = \pm\, 0.765 \ .$$

Since we want $Im(\lambda_c) < 0$, then $q = -0.765$. Using this result, the two conditions $\cos p \cosh q > 0$ and $\sin p \sinh q > 0$ imply that $\cos p > 0$ and $\sin p < 0$. This means that p lies in the fourth quadrant and so $p = -0.999$. Finally

$$\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = -0.999 - 0.765\, i \quad ,$$

$$\frac{\lambda_c}{b} = 0.572 - 0.765\, i \quad .$$

In the other case where $\cos\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ we obtain

$$\cos p \cosh q - i \sin p \sinh q = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \quad .$$

Hence: $$\cos p \cosh q = \frac{-1}{\sqrt{2}} = \sin p \sinh q \quad .$$

This leads to the same possibilities for p and q and again $\mathrm{Im}(\lambda_c) < 0$ requires $q = -0.765$. However in this case the relations $\sin p \sinh q < 0$ and $\cos p \cosh q < 0$ give that p lies in the second quadrant and so $p = \pi - 0.999$. Thus:

$$\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \pi - 0.999 - 0.765\, i,$$

$$\frac{\lambda_c}{b} = \pi + 0.572 - 0.765\, i \; .$$

We can now obtain the position z_c of the corner. We can write that

$$\tan\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \pm\left[\sec^2\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) - 1\right]^{\frac{1}{2}} = \pm\,(i-1)^{\frac{1}{2}} = \pm\,\sqrt[4]{2}\, e^{\frac{3\pi i}{4}} \; .$$

That is $$\tan\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = \pm\,(0.455 + 1.099i) \; .$$

Since $\mathrm{Im}(\lambda_c) < 0$, then $\mathrm{Im}[\tan(\frac{\lambda_c}{b} - \frac{\pi}{2})] < 0$ and so

$$\tan\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = -0.455 - 1.099i \; .$$

If we use this and the two values of $\frac{\lambda_c}{b}$ in equation (5.40) we get

$$z_c = (-1.220 - 1.671\, i)b \quad ,$$

and $$z_c = (-1.220 - (1.671 + \pi)i)b \; .$$

We must choose the highest value and so

$$z_c = -b\,(1.220 + 1.671\, i) \; .$$

This corresponds to

$$\lambda_c = b(0.572 - 0.765\, i)$$

and $$\tan\left(\frac{\lambda_c}{b} - \frac{\pi}{2}\right) = -\,\sqrt[4]{2}\;\, e^{\frac{3\pi i}{4}} \quad .$$

To calculate the angle c in terms of F we proceed as follows. We have $\sigma(\lambda) = -\sec^2(\frac{\lambda}{b} - \frac{\pi}{2})$. Hence

$$\sigma'(\lambda) = -\frac{2}{b}\sec^2(\frac{\lambda}{b} - \frac{\pi}{2})\tan(\frac{\lambda}{b} - \frac{\pi}{2}) \quad ,$$

and so using results stated above

$$-\sigma'(\lambda_c) = \frac{2}{b}(i)\sqrt[4]{2}\; e^{\frac{3\pi i}{8}} = \frac{2^{5/4}}{b}\; e^{\frac{7\pi i}{8}} \quad .$$

This gives

$$\mu = \arg(-\sigma'(\lambda_c)) = \frac{7\pi}{8} \quad .$$

In addition

$$\nu = \arg[U^2 + 2g\lambda_c]$$

$$= \arg[U^2 + 2gb(\pi - 0.999 - 0.765i)]$$

$$= \tan^{-1}\left[\frac{-0.765F}{1+2.143F}\right]$$

where $F = \frac{2gb}{U^2}$. Finally, then, using equation (5.06),

$$c = \frac{\pi}{3} - \frac{1}{3}\left[\frac{7\pi}{8} - 2\tan^{-1}\left[\frac{-0.765F}{1+2.143F}\right]\right]$$

$$= \frac{\pi}{24} - \frac{2}{3}\tan^{-1}\left[\frac{0.765F}{1+2.143F}\right] \quad .$$

Here, $c = \frac{\pi}{24}$ when $F = 0$, and $c \to -0.098 = -(5^\circ 37')$ when $F \to \infty$. It is clearly impossible to choose F so that $c = \frac{\pi}{3}$, i.e., so that the bed enters the corner horizontally.

Example 6 $\quad \sigma(\lambda) = -\sqrt{\frac{b}{\lambda}}$.

This example differs from the above in that $\lambda = 0$ corresponds to a finite point and that the flow at that point is not uniform. We have

$$-\int_c^{\lambda} \sigma(\lambda)d\lambda = 2\sqrt{b\lambda}\Big|_c^{\lambda} = 2\sqrt{b\lambda}$$

if we choose $c = 0$. Hence

$$z = -i\lambda + 2\sqrt{b\lambda} \quad .$$

The free streamline is

$$x = 2\sqrt{b\lambda}$$

which is a portion of a parabola opening down in the fourth quadrant. A sketch is given in Figure 5.13.

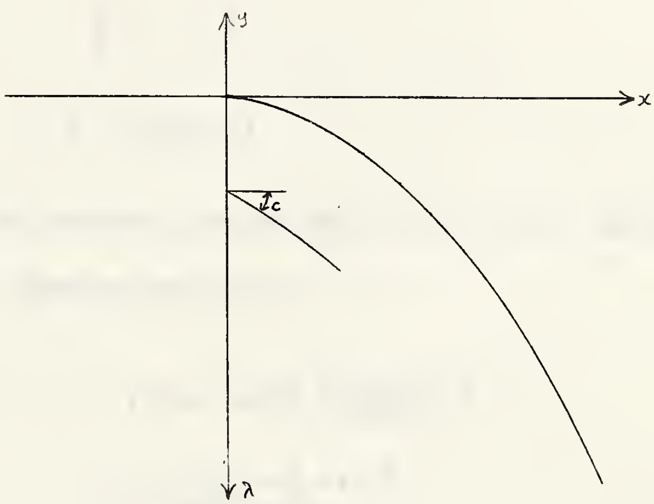


Figure 5.13

The corner occurs at λ_c where

$$\sigma(\lambda_c) = -\frac{b}{\lambda_c} = -i \quad .$$

This gives $\sqrt{\lambda c} = -i\sqrt{b}$,

$$\lambda_c = -b \quad .$$

This is an example of a flow where it is possible that $U^2 + 2g\lambda_c = 0$.

The position of the corner is

$$z_c = -i(-b) + 2(-ib) = -ib \quad .$$

Also, $\sigma'(\lambda) = \frac{\sqrt{b}}{2}\lambda^{-3/2}$, and so

$$\mu = \arg(-\sigma'(\lambda_c)) = \arg\left[\frac{-\sqrt{b}}{2\sqrt{\lambda_c}\,\lambda_c}\right] = \frac{\pi}{2} \quad .$$

In addition

$$\nu = \arg(U^2 + 2g\lambda_c) = \arg(U^2 - 2gb) = \arg(1-F) \quad .$$

In this example, three cases present themselves. We shall want to examine the velocity on the negative y-axis between the origin and $z_c = -ib$, and to this end we make the change of variable

$$\frac{\lambda}{b} = -i\,\xi \quad .$$

This gives $\quad z = ib\xi(\xi-2)$

which takes on the required values when $0 < \xi < 1$. Moreover, if we use equation (5.01), then we can write

$$\omega = (U^2 + 2g\lambda)^{\frac{1}{2}}\left(\frac{\sigma(\lambda)-i}{\sigma(\lambda)+i}\right)^{\frac{1}{2}}$$

$$= U(1+F\frac{\lambda}{b})^{\frac{1}{2}}\left(\frac{-\frac{b}{\lambda}-i}{\frac{b}{\lambda}+i}\right)^{\frac{1}{2}}$$

$$= U(1-\xi^2F)^{\frac{1}{2}}\left(\frac{1+\xi}{1-\xi}\right)^{\frac{1}{2}} \quad .$$

Case (a). $F < 1$.

In this case $\nu = \arg(1-F) = 0$ and so

$$c = \frac{\pi}{3} - \frac{1}{3}(\mu-2\nu) = \frac{\pi}{6} \quad .$$

Moreover, ω is real for all ξ in the range $0 < \xi < 1$, with, of course, a singularity at $\xi = 1$. Hence the velocity is horizontal on the negative

y-axis between the origin and the point $z_c = -ib$. We could thus continue the flow into the third quadrant by reflection in the y-axis and reversal of the direction of flow. This would create a symmetrical flow over a sharp corner of interior angle $\frac{2\pi}{3}$ with a parabolic free streamline.

Case (b). $F = 1$.

This is the case where $U^2 + 2g\lambda_c = 0$. Equation (5.09) gives $\mu_1 = \frac{\pi}{2}$, and equation (5.12) gives, with $\gamma = 1$,

$$c = \frac{\gamma+1}{8}\pi - \frac{3-\gamma}{4}\pi = 0 .$$

In this case

$$\omega = U(1-\xi^2)^{\frac{1}{2}}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{1}{2}} = U(1-\xi) ,$$

and so the velocity is again horizontal for $0 < \xi < 1$. However, in this case a stagnation point occurs at $z_c = -ib$. Again we can continue the flow into the third quadrant by reflection.

Case (c). $F > 1$.

Here the velocity is horizontal if $0 < \xi < \frac{1}{\sqrt{F}}$ and is vertical if $\xi > \frac{1}{\sqrt{F}}$. The point $\xi = \frac{1}{\sqrt{F}}$ is a stagnation point and so the flow cannot be analytically continued past it. This singularity corresponds to

$$z = \frac{-ib}{F}(2\sqrt{F} - 1) .$$

It corresponds to a λ value given by

$$\sqrt{\lambda} = -i\frac{U}{\sqrt{2g}} .$$

Hence $U^2 + 2g\lambda = 0$ for this singularity and the situation is dealt with in the material leading to equation (5.12). The situation here has

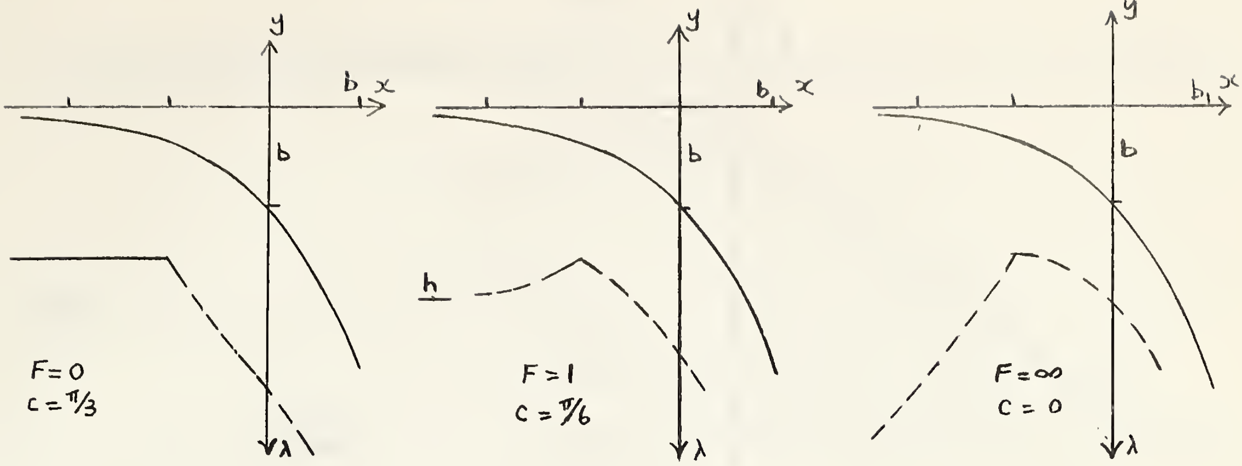
$$\sqrt{\lambda_c} = -\frac{iU}{\sqrt{2g}}$$

which gives $\mu_1 = \arg(-\sigma'(\lambda_c)) = \frac{\pi}{2}$.

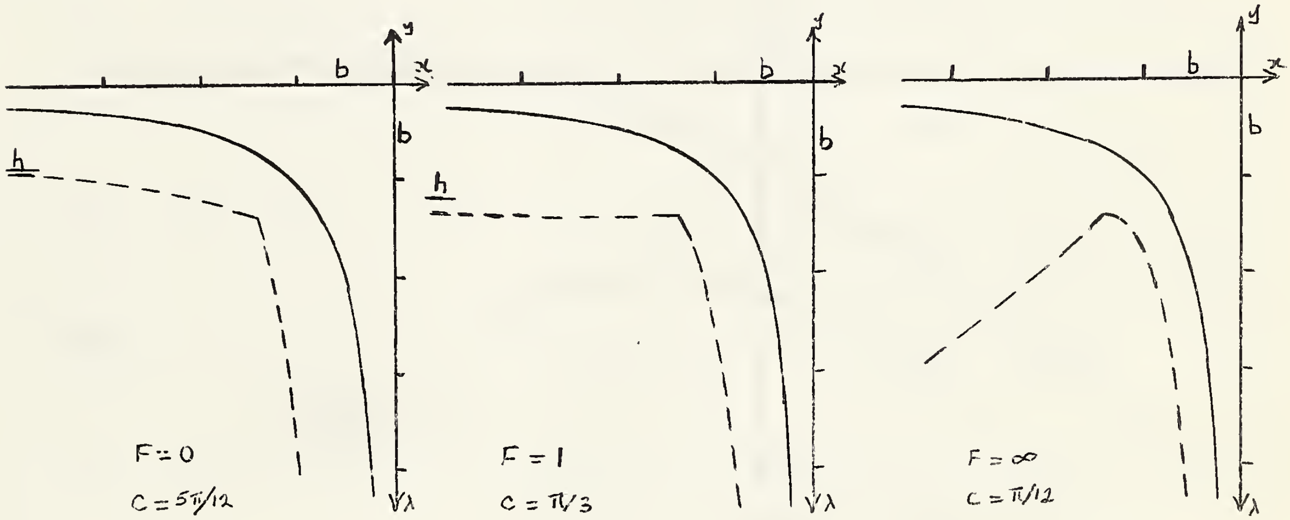
Since $\gamma = 1$, then equation (5.12) gives

$$c = 0$$

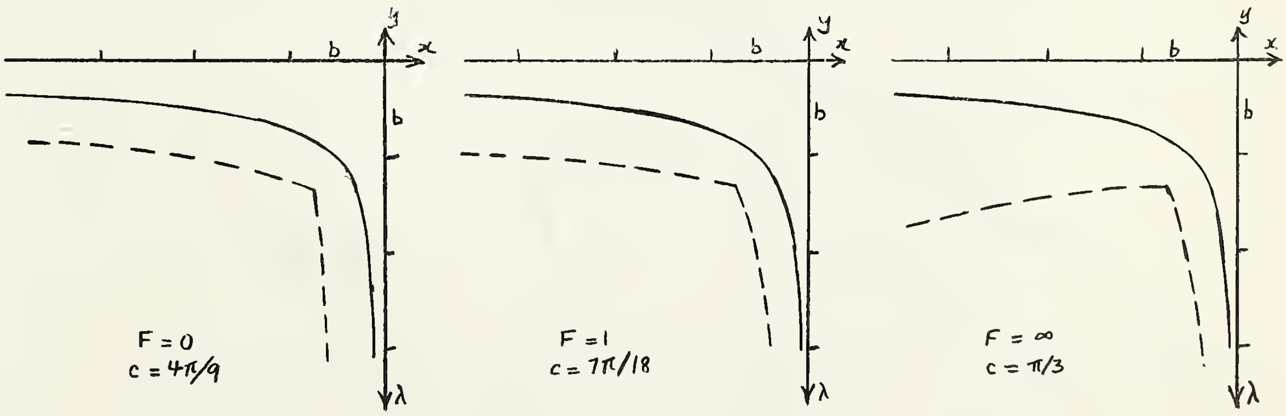
as in Case (b) above. Again the flow can be reflected in the y-axis.



Alpha = 1



Alpha = 2



Alpha = 3

Figure 5.14

Example 1 $\sigma = -b^{\alpha}/\lambda^{\alpha}$

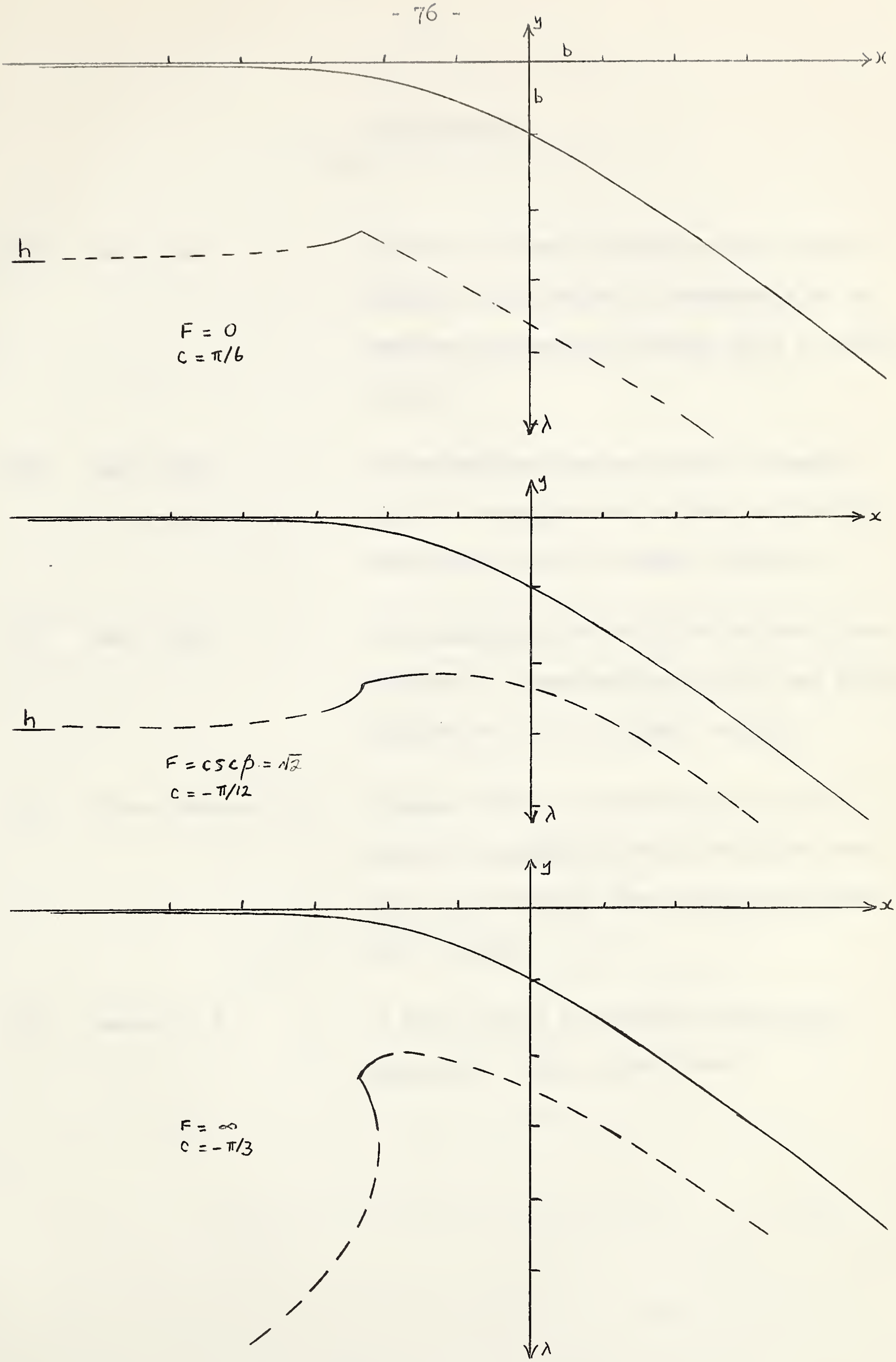


Figure 5.15

Example 2 $\sigma = -\cot\beta - b/\lambda$, $\beta = \pi/4$

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APPENDIX ONE

A THEOREM ON HARMONIC FUNCTIONS

<u>Theorem</u>: Given two functions $u(x,y)$ and $v(x,y)$ which are conjugate harmonics in the open region $y < 0$. If $u(x,y)$, $v(x,y)$ and $\frac{\partial u}{\partial x}$ exist and are continuous in the closed region $y \leq 0$, then $\frac{\partial v}{\partial y}$ exists and equals $\frac{\partial u}{\partial x}$ in the closed region $y \leq 0$.

<u>Proof</u>: The theorem is true in the open region since u and v are conjugate harmonics there. Let $(x_0,0)$ be on the boundary and let $k < 0$ be a real quantity. Then $v(x,y)$ exists and is continuous in the region $x = x_0$, $k \leq y \leq 0$, and $\frac{\partial v}{\partial x}$ exists and equals $\frac{\partial u}{\partial y}$ in the open region $x = x_0$, $k \leq y < 0$. Hence on the line $x = x_0$ the mean value theorem gives

$$v(x_0,k) - v(x_0,0) = k \left.\frac{\partial v}{\partial y}\right|_{(x_0,\xi)} ,$$

where $k < \xi < 0$. This is valid for all $k < 0$ and so

$$\lim_{k \to 0}\left[\frac{v(x_0,k)-v(x_0,0)}{k}\right] = \lim_{k \to 0}\left[\left.\frac{\partial v}{\partial y}\right|_{(x_0,\xi)}\right] .$$

The limit on the left is $\left.\frac{\partial v}{\partial y}\right|_{(x_0,0)}$ if it exists. We know that inside the fluid $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ by hypothesis. Hence, since $\xi \to 0$ as $k \to 0$, we have

$$\lim_{k \to 0}\left[\frac{v(x_0,k)-v(x_0,0)}{k}\right] = \lim_{\xi \to 0}\left[\left.\frac{\partial u}{\partial x}\right|_{(x_0,\xi)}\right] = \left.\frac{\partial u}{\partial x}\right|_{(x_0,0)}$$

since $\frac{\partial u}{\partial x}$ is continuous up to $y = 0$ by hypothesis. Hence the limit on the left exists and equals $\left.\frac{\partial u}{\partial x}\right|_{(x_0,0)}$ in the region $k \leq y \leq 0$.

APPENDIX TWO

THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

Lemma: If $g(x,\mu)$ is continuous in the closed bounded region R given by $|x-x_0| \le M$ and $|\mu-\mu_0| \le N$, then

$$f(x,\mu) \equiv \int_{x_0}^{x} g(t,\mu)dt$$

is uniformly continuous in R.

Proof: Since $g(x,\mu)$ is continuous in the bounded closed region R, then it is bounded and uniformly continuous there. The first of these assertions means that $|g(x,\mu)| < k$ in R; the second means that, given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|g(x,\mu) - g(x',\mu')| < \epsilon$$

for any two points (x,μ) and (x',μ') in R satisfying the conditions

$$|x-x'| < \delta \qquad \text{and} \qquad |\mu-\mu'| < \delta \quad .$$

We can write

$$f(x,\mu)-f(x',\mu') = \int_{x_0}^{x} [g(t,\mu)-g(t,\mu')]dt - \int_{x}^{x'} g(t,\mu')dt \quad .$$

If $|\mu-\mu'| < \delta$ this means

$$\begin{aligned} |f(x,\mu)-f(x',\mu')| &< \epsilon|x-x_0| + |x'-x|k \\ &< \epsilon M + \delta k \quad . \end{aligned}$$

Since this can be made arbitrarily small, the result is proved.

Using this we can prove the following two theorems. The first one is a generalization of section 5.6 and 5.7 of [5].

__Theorem 1:__ Given functions $f(x,y,\mu)$ and $y_0(\mu)$ which are continuous in their arguments in the region R of x-y-μ space given by

$$|x-x_0| \leq h, \quad |y-y_0| \leq Mh \quad \text{and} \quad |\mu-\mu_0| \leq k \ . \tag{A.01}$$

Suppose that, in the region R, $f(x,y,\mu)$ satisfies

$$|f(x,y,\mu)| < M \tag{A.02}$$

and

$$|f(x,y,\mu)-f(x,y^*,\mu)| \leq N|y-y^*| \ . \tag{A.03}$$

Then the solution, $y(x,\mu)$ of the differential equation

$$\frac{dy}{dx} = f(x,y,\mu) \tag{A.04}$$

which satisfies the boundary condition

$$y(x_0,\mu) = y_0(\mu) \tag{A.05}$$

exists, is unique, and is uniformly continuous in its arguments in the region R. Condition (A.03) is met if $\frac{\partial f}{\partial y}$ is bounded in R.

__Proof:__

(1) __Existence:__ If $y(x,\mu)$ satisfies equation (A.04) then a quadrature gives

$$y(x,\mu) = y_0(\mu) + \int_{x_0}^{x} f[x,y(x,\mu),\mu]dx \tag{A.06}$$

as a function which satisfies both (A.04) and (A.05). We now define a set of "successive approximations" to $y(x,\mu)$ by

$$y_0(x,\mu) = y_0(\mu) \quad , \tag{A.07}$$

$$y_n(x,\mu) = y_0(\mu) + \int_{x_0}^{x} f[x,y_{n-1}(x,\mu),\mu]dx \quad . \tag{A.08}$$

We will show that the sequence of functions $y_n(x,\mu)$ defined in this way converges uniformly to a unique function $y(x,\mu)$ satisfying (A.04) and (A.05). We use the notation $y_n = y_n(x,\mu)$.

We first show that, for all n,

$$|y_n - y_0| \leq Mh \quad . \tag{A.09}$$

This is obvious for $n = 0$ by equation (A.07) and holds for $n = 1$ by conditions (A.01) and (A.02). The proof proceeds by induction on n. If equation (A.09) is true for $n - 1$, then equation (A.01) gives that y_{n-1} is contained in R and so $|f(x,y_{n-1},\mu)| < M$ by (A.02). Hence (A.08) gives

$$|y_n - y_0| = \left| \int_{x_0}^{x} f(x,y_{n-1},\mu)dx \right| \leq M|x - x_0| \quad .$$

This, with (A.01), establishes (A.09).

We now show, again by induction, that

$$|y_n - y_{n-1}| < MN^{n-1} \frac{|x-x_0|^n}{n!} \quad . \tag{A.10}$$

If $n = 1$ this follows from (A.01) and (A.09). Assume it true for $n = p$. Now, (A.09) shows that y_p and y_{p-1} are contained in R and so (A.03) and the induction hypothesis imply

$$|f(x,y_p,\mu)-f(x,y_{p-1},\mu)| < N|y_p-y_{p-1}| < MN^p \frac{|x-x_0|^p}{p!} \quad .$$

Using this with equation (A.08) we have

$$|y_{p+1}-y_p| = \left| \int_{x_0}^{x} [f(x,y_p,\mu)-f(x,y_{p-1},\mu)]dx \right|$$

$$< \frac{MN^p}{p!} \int_{x_0}^{x} |x-x_0|^p dx$$

$$= \frac{MN^p}{(p+1)!} |x-x_0|^{p+1} \quad ,$$

which establishes (A.10).

If we now use (A.01) with (A.10), we obtain

$$|y_n-y_{n-1}| < \frac{M}{N} \frac{(Nh)^n}{n!}$$

which implies that the series

$$y_0 + \sum_{n=1}^{\infty} [y_n-y_{n-1}] \tag{A.11}$$

converges uniformly to some function $y = y(x,\mu)$ in the region R. It remains to show that $y(x,\mu)$ is a solution of the system (A.04) and (A.05).

Since the sum of p terms of the series (A.11) is y_{p-1}, then we have

$$\lim_{p\to\infty} [y_{p-1}] = y(x,\mu) \ . \tag{A.12}$$

This and (A.09) ensure that $|y(x,\mu)-y_0| < Mh$ and so $y(x,\mu)$ is in the region R. If we write $y = y(x,\mu)$ then equation (A.03) gives

$$|f(x,y,\mu) - f(x,y_{n-1},\mu)| \leq N|y-y_{n-1}| \quad .$$

Using (A.08) we can write

$$y-y_0-\int_{x_0}^{x} f(x,y,\mu)dx = y-[y_n-\int_{x_0}^{x} f(x,y_{n-1},\mu)dx] - \int_{x_0}^{x} f(x,y,\mu)dx$$

$$= y-y_n-\int_{x_0}^{x}[f(x,y,\mu)-f(x,y_{n-1},\mu)]dx \quad .$$

Finally, then

$$|y-y_0-\int_{x_0}^{x} f(x,y,\mu)dx| \leq |y-y_n| + N|y-y_{n-1}|h$$

using (A.01). Equation (A.12) implies that the right side can be made arbitrarily small if n is large enough. Hence the left side must be zero and so the function $y = y(x,\mu)$ satisfies

$$y(x,\mu) = y_0(\mu) + \int_{x_0}^{x} f[x,y(x,\mu),\mu]dx \quad .$$

It is easy to verify that this satisfies (A.04) and (A.05) and so the existence of a solution to this system has been proved.

(2) <u>Uniqueness</u>: To prove uniqueness, we suppose that $y(x,\mu)$ and $\eta(x,\mu)$ both satisfy the equations (A.04) and (A.05) where the conditions (A.02) and (A.03) are in effect. These functions both satisfy equation (A.06) and so we can write

$$y-\eta = \int_{x_0}^{x} [f(x,y,\mu)-f(x,\eta,\mu)]dx \ .$$

In addition, (A.06) implies that the points (x,y,μ) and (x,η,μ) are contained in R, and so (A.03) yields

$$|f(x,y,\mu)-f(x,\eta,\mu)| \leq N|y-\eta| \ .$$

These results give

$$|y-\eta| \leq N \int_{x_0}^{x} |y-\eta|\,dx \ . \tag{A.12}$$

If the function $e^{-Nx}\int_{x_0}^{x} |y-\eta|\,dx$ is differentiated with respect to x, then equation (A.12) shows that it does not increase with x. Since the function is zero when $x = x_0$ and is non-negative then it must be identically zero. This implies that $y(x,\mu) \equiv \eta(x,\mu)$ in R, which proves uniqueness.

(3) Uniform Continuity: Since $y_0(\mu)$ is continuous in the closed bounded region R, it is uniformly continuous there. Moreover, since f is continuous, then $f(x,y_0(\mu),\mu)$ is a continuous function of x and μ in R and so the lemma and (A.08) imply that $y_1(x,\mu)$ is uniformly continuous in R. This result for $y_n(x,\mu)$ follows by induction on n. Finally, the uniform convergence of the series (A.11) gives the result that $y(x,\mu)$ is uniformly continuous in R.

Theorem 2: If $H(G,z)$ is an analytic function of its complex arguments in the region R of $G-z$ space given by

$$|z - z_0| < h \ , \quad |G - G_0| < Mh \ ,$$

and if conditions $|H(G,z)| < M$ and $|H(G,z) - H(G^*,z)| \leq N|G-G^*|$ are imposed on H in R, then the solution $G(z)$ of the differential equation $\frac{dG}{dz} = H(G,z)$ which satisfies the condition $G(z_0) = G_0$ exists and is a unique analytic function of the complex variable z in the region R.

Proof: Since R is an open region, here no question of uniform continuity arises. The proofs of existence and uniqueness go through as in Theorem 1 with x and y replaced by z and G respectively and with the appropriate replacement of $\leq$ with $<$ due to the different definition of R. It is assumed that paths of integration, where they occur lie inside R.

We now prove the analyticity of $G(z)$. The equations corresponding to (A.07) and (A.08) are

$$G_0(z) = G_0$$

and

$$G_n(z) = G_0 + \int_{z_0}^{z} H[G_{n-1}(z), z]dz \quad .$$

Since the integral of an analytic function is analytic, then $G_1(z)$ and, by induction on n, $G_n(z)$ are analytic in R. Again the uniform convergence of the series (A.11) ensures the analyticity of $G(z)$.

These results are valid when the boundary condition is satisfied at a point y_0 inside the region R. They also hold if the condition is met at a point on the boundary as is shown in Graves, "Theory of Functions of Real Variables, Second Edition", McGraw-Hill, 1956, page 157.